Unique continuation results for Ricci curvature and applications

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Abstract

Unique continuation results are proved for metrics with prescribed Ricci curvature in the setting of bounded metrics on compact manifolds with boundary, and in the setting of complete conformally compact metrics on such manifolds. Related to this issue, an isometry extension property is proved: continuous groups of isometries at conformal infinity extend into the bulk of any complete conformally compact Einstein metric. Relations of this property with the invariance of the Gauss–Codazzi constraint equations under deformations are also discussed.
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1. Introduction

In this paper, we study certain issues related to the boundary behavior of metrics with prescribed Ricci curvature. Let M be a compact $(n + 1)$-dimensional manifold with compact non-empty boundary ∂M. We consider two possible classes of Riemannian metrics g on M. First, g may extend smoothly to a Riemannian metric on the closure $\bar{M} = M \cup \partial M$, thus inducing a Riemannian metric $\gamma = g|_{\partial M}$ on ∂M. Second, g may be a complete metric on M, so that ∂M is "at infinity". In this case, we assume that g is conformally compact, i.e. there exists a defining function ρ for ∂M in M such that the conformally equivalent metric

$$\widetilde{g} = \rho^2 g \tag{1.1}$$

extends at least C^2 to ∂M. The defining function ρ is unique only up to multiplication by positive functions; hence only the conformal class $[\gamma]$ of the associated boundary metric $\gamma = \bar{g}|_{\partial M}$ is determined by (M, g).

The issue of boundary regularity of Riemannian metrics g with controlled Ricci curvature has been addressed recently in several papers. Thus, [4] proves boundary regularity for bounded metrics g on M with controlled Ricci curvature, assuming control on the boundary metric γ and the mean curvature of ∂M in M. In [16], boundary regularity

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is proved for conformally compact Einstein metrics with smooth conformal infinity; this was previously proved by different methods in dimension 4 in [3], cf. also [5].

One purpose of this paper is to prove a unique continuation property at the boundary ∂M for bounded metrics or for conformally compact metrics. We first state a version of the result for Einstein metrics on bounded domains.

Theorem 1.1. *Let (M, g) be a $C^{3,\alpha}$ metric on a compact manifold with boundary M, with induced metric $\gamma = g|_{\partial M}$, and let A be the 2nd fundamental form of ∂M in M. Suppose the Ricci curvature Ric_g satisfies*

$$\text{Ric}_g = \lambda g, \tag{1.2}$$

where λ is a fixed constant.

Then (M, g) is uniquely determined up to local isometry and inclusion, by the Cauchy data (γ, A) on an arbitrary open set U of ∂M.

Thus, if (M_1, g_1) and (M_2, g_2) are a pair of Einstein metrics as above, whose Cauchy data (γ, A) agree on an open set U common to both ∂M_1 and ∂M_2, then after passing to suitable covering spaces $\bar{M}_i$, either there exist isometric embeddings $\bar{M}_1 \subset \bar{M}_2$ or $\bar{M}_2 \subset \bar{M}_1$ or there exists an Einstein metric $(\bar{M}_3, g_3)$ and isometric embeddings $(\bar{M}_i, g_i) \subset (\bar{M}_3, g_3)$. Similar results hold for metrics which satisfy other covariant equations involving the metric to second order, for example the Einstein equations coupled to other fields; see Proposition 3.7.

For conformally compact metrics, the second fundamental form A of the compactified metric $\bar{g}$ in (1.1) is umbilic, and completely determined by the defining function ρ. In fact, for conformally compact Einstein metrics, the higher order Lie derivatives $\mathcal{L}_N^{(k)} \bar{g}$ at ∂M, where N is the unit vector in the direction $\bar{\nabla}\rho$, are determined by the conformal infinity $[\gamma]$ and ρ up to order $k < n$. Supposing ρ is a geodesic defining function, so that $\|\bar{\nabla}\rho\| = 1$, let

$$g_{(n)} = \frac{1}{n!} \mathcal{L}_N^{(n)} \bar{g}. \tag{1.3}$$

More precisely, $g_{(n)}$ is the nth term in the Fefferman–Graham expansion of the metric g; this is given by (1.3) when n is odd, and in a similar way when n is even, cf. [18] and Section 4. The term $g_{(n)}$ is the natural analog of A for conformally compact Einstein metrics.

Theorem 1.2. *Let g be a C^2 conformally compact Einstein metric on a compact manifold M with C^∞ smooth conformal infinity $[\gamma]$, normalized so that*

$$\text{Ric}_g = -ng. \tag{1.4}$$

Then the Cauchy data $(\gamma, g_{(n)})$ restricted to any open set U of ∂M uniquely determine (M, g) up to local isometry and determine $(\gamma, g_{(n)})$ globally on ∂M.

The recent boundary regularity result of Chruściel et al. [16], implies that (M, g) is C^∞ polyhomogeneous conformally compact, so that the hypotheses of Theorem 1.2 imply the term $g_{(n)}$ is well defined on ∂M. A more general version of Theorem 1.2, without the smoothness assumption on $[\gamma]$, is proved in Section 4, cf. Theorem 4.1. For conformally compact metrics coupled to other fields, see Remark 4.5.

Of course neither Theorem 1.1 or 1.2 hold when just the boundary metric γ on $U \subset \partial M$ is fixed. For example, in the context of Theorem 1.2, by [20,16], given any C^∞ smooth boundary metric γ sufficiently close to the round metric on S^n, there is a smooth (in the polyhomogeneous sense) conformally compact Einstein metric on the $(n + 1)$-ball B^{n+1}, close to the Poincaré metric. Hence, the behavior of γ in U is independent of its behavior on the complement of U in ∂M.

Theorems 1.1 and 1.2 have been phrased in the context of "global" Einstein metrics, defined on compact manifolds with compact boundary. However, the proofs are local, and these results hold for metrics defined on an open manifold with boundary. From this perspective, the data (γ, A) or $(\gamma, g_{(n)})$ on U determine whether Einstein metric g has a global extension to an Einstein metric on a compact manifold with boundary (or conformally compact Einstein metric), and how smooth that extension is at the global boundary.

A second purpose of the paper is to prove the following isometry extension result which is at least conceptually closely related to Theorem 1.2. However, while Theorem 1.2 is valid locally, this result depends crucially on global properties.

Theorem 1.3. *Let g be a C^2 conformally compact Einstein metric on a compact manifold M with C^∞ boundary metric $(\partial M, \gamma)$, and suppose*

$$\pi_1(M, \partial M) = 0. \tag{1.5}$$

Then any connected group of isometries of $(\partial M, \gamma)$ extends to an action by isometries on (M, g).

The condition (1.5) is equivalent to the statement that ∂M is connected and the inclusion map $\iota : \partial M \to M$ induces a surjection $\pi_1(\partial M) \to \pi_1(M) \to 0$.

Rather surprisingly, this result is closely related to the equations at conformal infinity induced by the Gauss–Codazzi equations on hypersurfaces tending to ∂M. It turns out that isometry extension from the boundary at least into a thickening of the boundary is equivalent to the requirement that the Gauss–Codazzi equations induced at ∂M are preserved under arbitrary deformations of the boundary metric. This is discussed in detail in Section 5, see e.g. Proposition 5.4. We note that this result does not hold for complete, asymptotically (locally) flat Einstein metrics, cf. Remark 5.8.

A simple consequence of Theorem 1.3 is the following uniqueness result:

Corollary 1.4. *A C^2 conformally compact Einstein metric with conformal infinity given by the class of the round metric g_{+1} on the sphere S^n is necessarily isometric to the Poincaré metric on the ball B^{n+1}.*

Results similar to Theorem 1.3 and Corollary 1.4 have previously been proved in a number of different special cases by several authors, see for example [7,9,31,33]; the proofs in all these cases are very different from the proof given here.

It is well known that unique continuation does not hold for large classes of elliptic systems of PDEs, even for general small perturbations of systems which are diagonal at leading order; see for instance [23] and references therein for a discussion related to geometric PDEs. The proofs of Theorems 1.1 and 1.2 rely on unique continuation results of Calderón [13,14] and Mazzeo [27] respectively, based on Carleman estimates. The main difficulty in reducing the proofs to these results is the diffeomorphism covariance of the Einstein equations and, more importantly, that of the "abstract" Cauchy data (γ, A) or $(\gamma, g_{(n)})$ at ∂M. The unique continuation theorem of Mazzeo requires a diagonal (i.e. uncoupled) Laplace-type system of equations, at leading (second) order. The unique continuation result of Calderón is more general, but again requires strong restrictions on the structure of the leading order symbol of the operator. For emphasis and clarity, these issues are discussed in more detail in Section 2. The proofs of Theorems 1.1–1.3 are then given in Sections 3–5 respectively.

Very recently, while the writing on this paper was being completed, Biquard [12] has given a different proof of Theorem 1.2, which avoids some of the gauge issues discussed above. However, his method apparently requires C^∞ smoothness of the boundary data, which limits the applicability of this result; for instance the applications in [5] or [6] require finite or low differentiability of the boundary data.

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2. Local coordinates and Cauchy data

In this section, we discuss in more detail the remarks in the Introduction on classes of local coordinate systems, and their relation with Cauchy data on the boundary ∂M.

Thus, consider for example solutions to the system

$$\text{Ric}_g = 0, \tag{2.1}$$

defined near the boundary ∂M of an $(n + 1)$-dimensional manifold M. Since the Ricci curvature involves two derivatives of the metric, Cauchy data at ∂M consist of the boundary metric γ and its first derivative, invariantly represented by the second fundamental form A of ∂M in M. Thus, we assume (γ, A) are prescribed at ∂M (subject to the Gauss and Gauss–Codazzi equations), and call (γ, A) abstract Cauchy data. Observe that the abstract Cauchy data are invariant under diffeomorphisms of M equal to the identity at ∂M.

The metric g determines the geodesic defining function

$$t(x) = \text{dist}_g(x, \partial M).$$

The function t depends of course on g; however, given any other smooth metric g', there is a diffeomorphism F of a neighborhood of ∂M, equal to the identity on ∂M, such that $t'(x) = \text{dist}_{F^*g'}(x, \partial M)$ satisfies $t' = t$. As noted above, this normalization does not change the abstract Cauchy data (γ, A) and preserves the isometry class of the metric.

Let $\{y^\alpha\}$, $0 \le \alpha \le n$, be any local coordinates on a domain Ω in M containing a domain U in ∂M. We assume that $\{y^i\}$ for $1 \le i \le n$ form local coordinates for ∂M when $y^0 = 0$, so that $\partial/\partial y^0$ is transverse to ∂M. Throughout the paper, Greek indices α, β run from 0 to n, while Latin indices i, j run from 1 to n. If $g_{\alpha\beta}$ are the components of g in these coordinates, then the abstract Cauchy problem associated to (2.1) in the local coordinates $\{y^\alpha\}$ is the system

$$(\text{Ric}_g)_{\alpha\beta} = 0, \quad \text{with } g_{ij}|_U = \gamma_{ij}, \frac{1}{2}(\mathcal{L}_{\nabla t}g)_{ij}|_U = a_{ij}, \tag{2.2}$$

where γ_{ij} and a_{ij} are given on U (subject to the constraints of the Gauss and Gauss–Codazzi equations). Here one immediately sees a problem, in that (2.2) on $U \subset \partial M$ involves only the tangential part g_{ij} of the metric (at 0 order), and not the full metric $g_{\alpha\beta}$ at U. The normal g_{00} and mixed g_{0i} components of the metric are not prescribed at U. As seen below, these components are gauge dependent; they cannot be prescribed "abstractly", independent of coordinates, as is the case with γ and A. In other words, if (2.1) is expressed in local coordinates $\{y^\alpha\}$ as above, then a well-defined Cauchy or unique continuation problem has the form

$$(\text{Ric}_g)_{\alpha\beta} = 0, \quad \text{with } g_{\alpha\beta} = \gamma_{\alpha\beta}, \quad \frac{1}{2}\partial_t g_{\alpha\beta} = a_{\alpha\beta}, \quad \text{on } U \subset \partial M, \tag{2.3}$$

where Ω is an open set in $(\mathbb{R}^{n+1})^+$ with $\partial\Omega = U$ an open set in $\partial(\mathbb{R}^{n+1})^+ = \mathbb{R}^n$. Formally, (2.3) is a determined system, while (2.2) is underdetermined.

Let g_0 and g_1 be two solutions to (2.1), with the same Cauchy data (γ, A), and with geodesic defining functions t_0, t_1. Changing the metric g_1 by a diffeomorphism if necessary, one may assume that $t_0 = t_1$. One may then write the metrics with respect to a Gaussian or geodesic boundary coordinate system (t, y^i) as

$$g_k = dt^2 + (g_k)_t, \tag{2.4}$$

where $(g_k)_t$ is a curve of metrics on ∂M and $k = 0, 1$. Here y_i are coordinates on ∂M which are extended into M to be invariant under the flow of the vector field ∇t. The metric $(g_k)_t$ is the metric induced on $S(t)$ and pulled back to ∂M by the flow of ∇t. One has $(g_k)_0 = \gamma$ and $\frac{1}{2}\frac{d}{dt}(g_k)_t|_{t=0} = A$. Since $g_{0\alpha} = \delta_{0\alpha}$ in these coordinates, $\nabla t = \partial_t$, and hence the local coordinates are the same for both metrics (or at least may be chosen to be the same). Thus, geodesic boundary coordinates are natural from the point of view of the Cauchy or unique continuation problem, since in such local coordinates the system (2.2), together with the prescription $g_{0\alpha} = \delta_{0\alpha}$, is equivalent to the system (2.3). However, the Ricci curvature is not elliptic or diagonal to leading order in these coordinates. The expression of the Ricci curvature in such coordinates does not satisfy the hypotheses of Calderón's theorem [14], and it appears to be difficult to establish unique continuation of solutions in these coordinates by working directly on the equations on the metric (see, however, [12] for another approach).

Next suppose that $\{x^\alpha\}$ are boundary harmonic coordinates, defined as follows. For $1 \le i \le n$, let $\hat{x}^i$ be local harmonic coordinates on a domain U in $(\partial M, \gamma)$. Extend $\hat{x}^i$ into M to be harmonic functions in (Ω, g), $\Omega \subset M$, with Dirichlet boundary data; thus

$$\Delta_g x^i = 0, \quad x^i|_U = \hat{x}^i. \tag{2.5}$$

Let x^0 be a harmonic function on Ω with 0 boundary data, so that

$$\Delta_g x^0 = 0, \quad x^0|_U = 0. \tag{2.6}$$

Then the collection $\{x^\alpha\}$, $0 \le \alpha \le n$, form a local harmonic coordinate chart on a domain $\Omega \subset (M, g)$. In such coordinates, one has

$$(\text{Ric}_g)_{\alpha\beta} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + Q_{\alpha\beta}(g, \partial g), \tag{2.7}$$

where $Q(g, \partial g)$ depends only on g and its first derivatives. This is an elliptic operator, diagonal at leading order, and satisfies the hypotheses of Calderón's theorem. However, in general, the local Cauchy problem (2.3) is not well defined in these coordinates; if g_0 and g_1 are two solutions of (2.1), each with corresponding local boundary harmonic coordinates, then the components $(g_0)_{0\alpha}$ and $(g_1)_{0\alpha}$ in general will differ at $U \subset \partial M$. This is of course closely related to the fact that there are many possible choices of harmonic functions x^α satisfying (2.5) and (2.6), and to the fact that the behavior of harmonic functions depends on global properties of (Ω, g). In any case, it is not known how to set up a well-defined Cauchy problem in these coordinates for which one can apply standard unique continuation results.

Consider then geodesic–harmonic coordinates "intermediate" between geodesic boundary and boundary harmonic coordinates. Thus, let t be the geodesic distance to ∂M as above. Choose local harmonic coordinates $\hat{x}^i$ on ∂M as before and extend them into M to be harmonic on the level sets $S(t)$ of t, i.e. locally on $S(t)$,

$$\Delta_{U(t)} x^i = 0, \quad x^i|_{\partial U(t)} = \hat{x}^i|_{\partial U(t)}; \tag{2.8}$$

here the boundary value $\hat{x}^i$ is the extension of $\hat{x}^i$ on U into M which is invariant under the flow ϕ_t of ∇t, and $U(t) = \phi_t(U) \subset S(t)$. The functions (t, x^i) form a coordinate system in a neighborhood Ω in M with $\Omega \cap \partial M = U$.

It is not difficult to prove that geodesic–harmonic coordinates preserve the Cauchy data, in the sense that the data (2.2) in such coordinates imply the data (2.3). However, the Ricci curvature is not an elliptic operator in the metric in these coordinates, nor is it diagonal at leading order; the main reason is that the mean curvature of the level sets $S(t)$ is not a priori controlled. So again, it remains an open question whether unique continuation can be proved in these coordinates.

Having listed these attempts which appear to fail, a natural choice of coordinates which do satisfy the necessary requirements are H-harmonic coordinates (τ, x^i), whose τ-level surfaces S_τ are of prescribed mean curvature H and with x^i harmonic on S_τ. These coordinates were introduced by Andersson–Moncrief [8] to prove a well-posedness result for the Cauchy problem for the Einstein equations in general relativity, and, as shown in [8], have a number of advantageous properties. Thus, adapting some of the arguments of [8], we show in Section 3 that the Einstein equations (1.2) are effectively elliptic in such coordinates, and such coordinates preserve the Cauchy data in the sense above (i.e. (2.2) implies (2.3)). It will then be shown that unique continuation holds in such coordinates, via application of the Calderón theorem.

3. Proof of Theorem 1.1

Theorem 1.1 follows from a purely local result, which we formulate as follows. Let C be a domain diffeomorphic to a cylinder $I \times B^n \subset \mathbb{R}^{n+1}$, with $U = \{0\} \times B^n$, diffeomorphic to a ball in $\mathbb{R}^n$. Let $U = \partial_0 C$ be the horizontal boundary and $\partial C = I \times S^{n-1}$ be the vertical boundary.

Let g be a Riemannian metric on C which is $C^{k-1,\alpha}$ up to the boundary of C in the given standard coordinate system $\{y^\alpha\} = \{y^0, y^i\}$ with $y^0 = 0$ on U and $k \geq 2$. Without loss of generality, we assume that C is chosen sufficiently small so that g is close to the Euclidean metric δ in the $C^{k-1,\alpha}$ topology. For simplicity, we shall rescale C and the coordinates $\{y^\alpha\}$ if necessary so that (C, g) is $C^{k-1,\alpha}$ close to the standard cylinder $((I \times B^n(1), B^n(1))) \subset (\mathbb{R}^{n+1}, \mathbb{R}^n)$, $I = [0, 1]$.

We will prove the following local version of Theorem 1.1.

Theorem 3.1. *Let g_0, g_1 be two $C^{k-1,\alpha}$ metrics as above on C, $k \geq 4$, satisfying*

$$\text{Ric}_{g_i} = \lambda g_i, \quad i = 0, 1 \tag{3.1}$$

for some fixed constant λ. Suppose g_0 and g_1 have the same abstract Cauchy data on U in the sense of Section 2, so that $\gamma_0 = \gamma_1$ and $A_0 = A_1$.

Then (C, g_0) is isometric to (C, g_1), by an isometry equal to the identity on U. In particular, Theorem 1.1 holds.

The proof of Theorem 3.1 will proceed in several steps, organized around several lemmas. We first work with a fixed metric g on C as above. Let N be the inward unit normal to U in C and let $A = \nabla N$ be the corresponding second fundamental form, with mean curvature $H = tr_g A$ on U. By the initial assumptions above, A and H are close to 0 in $C^{k-2,\alpha}$; more precisely, one may assume that

$$\|A\|_{C^{k-2,\alpha}} = O(\varepsilon)$$

with ε positive but as small as needed, by a further rescaling of the coordinates (this will play an important role at various places below). Note moreover that the rescaling process turns the Einstein constant λ into $\varepsilon\lambda$. Abusing notation here, we denote $y^0 = t$ and without loss of generality assume that the coordinates y^i are harmonic on U.

To begin, we construct certain systems of H-harmonic coordinates discussed at the end of Section 2. Let $\phi : C \to C$ be a diffeomorphism of the cylinder C (in other words a change of coordinates), so that $y^\alpha = \phi^\alpha(x^\beta)$, where x^β is another coordinate system for C. As above, we write $x^\alpha = (\tau, x^i)$ and assume that ϕ is close to the identity map. The level surfaces $S_\tau = \{\tau\} \times B^n$ are mapped under ϕ to a foliation Σ_τ of C, with each leaf given by the graph of the function ϕ_τ over B^n. We assume $\phi_0 = id$, so that $\phi = id$ on U. Let $f : \partial C \to \partial C$ be the induced diffeomorphism on the boundary ∂C.

Lemma 3.2. *Let $k \geq 2$. Given a $C^{k,\alpha}$ mapping f on ∂C as above, close to the identity in $C^{k,\alpha}$, and a metric g close to the Euclidean metric δ in $C^{k-1,\alpha}$ on C, there exists a unique $\phi \in \mathrm{Diff}^{k,\alpha}(C)$ such that, with respect to the pull-back metric $\phi^*(g)$,*

$$H^{\phi^*(g)}(S_\tau) = H^{\phi^*(g)}(S_0), \quad and \quad \Delta_{S_\tau}^{\phi^*(g)} x^i = 0, \tag{3.2}$$

with the property that $\phi_{|\partial C} = f$. Thus, the leaves $\tau = const$ have mean curvature independent of τ, in the x^α-coordinates, and the coordinate functions x^i are harmonic on each S_τ.

Proof. Let

$$\mathcal{H} : \mathrm{Met}^{k-1,\alpha}(C) \times \mathrm{Diff}_0^{k,\alpha}(C) \longrightarrow C^{k-2,\alpha}(C) \times \prod_1^n C^{k-2,\alpha}(C) \times \mathrm{Diff}_0^{k,\alpha}(\partial C)$$

$$\mathcal{H}(g, F) = (H^{F^*(g)}(S_\tau) - H^{F^*(g)}(S_0), \Delta_{S_\tau}^{F^*(g)} x^i, F_{|\partial C}),$$

where $\mathrm{Diff}_0^{k,\alpha}(C)$ is the space of $C^{k,\alpha}$ diffeomorphisms on the cylinder equal to the identity on $C_0 = \{0\} \times B^n$. The map $\mathcal{H}$ is clearly a smooth map of Banach spaces, and its linearization at (δ, id) in the second variable is

$$L(v) = (\Delta_\delta v^0, \Delta_\delta v^i, v_{|\partial C}),$$

where Δ_δ is the Laplacian with respect to the flat metric δ on S_τ. The operator L is clearly an isomorphism, and by the implicit function theorem in Banach spaces, it follows that there is a smooth map

$$\Phi : \mathcal{U} \times \mathcal{V} \subset \mathrm{Met}^{k-1,\alpha}(C) \times \mathrm{Diff}_0^{k,\alpha}(\partial C) \longrightarrow \mathrm{Diff}_0^{k,\alpha}(C),$$
$$\Phi(g, f) = \phi^g(f)$$

from a neighborhood of the Euclidean metric and the identity map such that $(\phi^g(f))_{|\partial C} = f$, and satisfying (3.2).

Note moreover that $\phi^g(f)$ is $C^{k,\alpha}$-close to the identity if f is close to it on ∂C and g is $C^{k-1,\alpha}$-close to the Euclidean metric on C. This implies that the family $\{\Sigma_\tau\}$ forms a $C^{k,\alpha}$ foliation of C. ∎

The metric g in the $x^\alpha = (\tau, x^i)$ coordinates, i.e. $\phi^* g$, may be written in lapse/shift form, commonly used in general relativity, as

$$g = u^2 d\tau^2 + g_{ij}(dx^i + \sigma^i d\tau)(dx^j + \sigma^j d\tau), \tag{3.3}$$

where u is the lapse and σ is the shift in the x-coordinates and g_{ij} is the induced metric on the leaves $S_\tau = \{\tau = \mathrm{const}\}$. A simple computation shows that lapse and shift are related to the metric $g = g_{\alpha\beta} dy^\alpha dy^\beta$ in the initial (y^α) coordinates by the equations

$$u^2 + |\sigma|^2 = g_{\alpha\beta}^y (\partial_\tau \phi^\alpha)(\partial_\tau \phi^\beta), \tag{3.4}$$

$$g_{ij}\sigma^j = g_{\alpha\beta}^y (\partial_\tau \phi^\alpha)(\partial_i \phi^\beta), \tag{3.5}$$

$$g_{ij} = g_{\alpha\beta}^y (\partial_i \phi^\alpha)(\partial_j \phi^\beta). \tag{3.6}$$

A computation using (3.5) shows that $|\sigma|^2 = g^{ij} g_{\alpha\beta}^y g_{\mu\nu}^y \partial_\tau \phi^\alpha \partial_\tau \phi^\mu \partial_i \phi^\beta \partial_j \phi^\nu$. From $g_{0j} = g_{ij}\sigma^i$ and $g_{00} = u^2 + |\sigma|^2$, one may compute $g^{\alpha\beta}$ and, expanding, this yields $g^{00} = u^{-2}$ and $\sigma^i = -u^2 g^{0i}$. The unit normal N to the foliation

Σ_τ is given by

$$N = u^{-1}(\partial_\tau - \sigma),\tag{3.7}$$

so that, for instance, $g(N, \cdot) = u\mathrm{d}\tau$ (this will be useful later on).

It is now important to notice that the construction of H-harmonic coordinates in Lemma 3.2 can be done for any choice of boundary diffeomorphism f. We shall show that there is a (unique) choice of f close to the identity with $f = id$ on $\partial_0 C = \{0\} \times S^{n-1}$, such that u is identically 1 and the shift σ vanishes on the vertical boundary $\partial C = I \times S^{n-1}$.

Lemma 3.3. *For any $k \geq 3$, there exists a $C^{k,\alpha}$ diffeomorphism $f : \partial C \to \partial C$ such that the lapse u and shift σ of g in (3.3) satisfies*

$$u = 1, \quad and \quad \sigma = 0, \quad on\ \partial C.\tag{3.8}$$

Proof. Consider the operator

$$\Xi : \mathrm{Met}^{k-1,\alpha}(C) \times \mathrm{Diff}_0^{k,\alpha}(\partial C) \to C^{k-1,\alpha}(\partial C) \times \prod_1^n C^{k-1,\alpha}(\partial C),\tag{3.9}$$

$$\Xi(g, f) = (g_{\alpha\beta}^y (\partial_\tau \phi^\alpha)(\partial_\tau \phi^\beta) - |\sigma|^2(\phi), \sigma^i(\phi)),$$

where $\phi = \phi^g(f)$ is defined above in the proof of Lemma 3.2; recall that $\phi|_{\partial C} = f$. More precisely, Ξ is defined in the neighborhoods $\mathcal{U}$ and $\mathcal{V}$ defined in Lemma 3.2. From (3.5), one has $\sigma^i = g^{ij} g_{\alpha\beta}^y \partial_\tau \phi^\alpha \partial_j \phi^\beta$. Note that for the map $f = id$ on ∂C, and at the metric $g_0 = \delta$, one has $\phi^{g_0}(id) = id$ and $|\xi|^2(id) = 0$, so that $\Xi(g_0, id) = (1, 0, 0)$. Thus,

$$\Xi(g, id) = (1 + O(\varepsilon), O(\varepsilon)),\tag{3.10}$$

where, as already discussed, ε is positive and may be taken as small as needed. We would like to apply the implicit function theorem to assert that for any $g \in \mathcal{U}$, where $\mathcal{U}$ is sufficiently small, there exists $f = f(g) \in \mathcal{V}$, such that

$$\Xi(g, f(g)) = (1, 0).\tag{3.11}$$

If such f exists, then, for any $g \in \mathcal{U}$, the pair (g, f) defines a $C^{k,\alpha}$ diffeomorphism $\phi : C \to C$ and the resulting metric $\phi^* g$ satisfies (3.8). Thus it suffices to solve (3.11).

There is however a loss of one derivative in the map Ξ and its derivative in the second variable, as is obvious by looking at its value at the metric $g_0 = \delta$:

$$(D_2\Xi)_{(g_0, id)}(h) = (2\partial_\tau h^0, \partial_\tau h^i + \partial_i h^0).\tag{3.12}$$

Thus, we need to use the Nash–Moser inverse function theorem. We use this in the form given in [34, Section 6.3], and in particular [34, Thm. 6.3.3, Cor. 1, Cor. 2]. Following Zehnder's notation (with s in place of σ), let $X_s = \mathrm{Met}^{k-1,\alpha}(C), Y_s = \mathrm{Diff}_0^{k,\alpha}(\partial C)$, and $Z_s = C^{k-1,\alpha}(\partial C) \times \prod_1^n C^{k-1,\alpha}(\partial C)$, so that s is a linear function of $k+\alpha$. Thus we write $X_s = \mathrm{Met}^{s+1+\varepsilon}(C)$, for some arbitrary but fixed $\varepsilon > 0$ (recall we start at $k \geq 2$), $Y_s = \mathrm{Diff}_0^{s+2+\varepsilon}(\partial C)$ and $Z_s = \prod_1^n C^{s+1+\varepsilon}(\partial C)$. We check the hypotheses of Zehnder's theorem:

(H1) When $s = 0$, Ξ is C^2 in f, with uniform bounds in Y_0. This is clearly true.
(H2) Ξ is Lipschitz in X_0, also true.
(H3) Ξ is of order $s = \infty$, with growth $\delta = 1$. This follows from

$$\|\Xi(g, f)\|_{C^{k-2,\alpha}} \leq C(k)(\|g\|_{C^{k-1,\alpha}} + \|f\|_{C^{k,\alpha}}).$$

(H4) Existence of right inverse of loss $\gamma = 1$. Let $(D_2\Xi)_{(g,f)}$ be the derivative of Ξ with respect to the second variable f at (g, f). Then varying f in the direction v, $f_s = f + sv$, it is easy to see that the operator $(D_2\Xi)_{(g,f)}$ is a first-order linear PDE in v, with all coefficients in $C^{k-1,\alpha}$. As in (3.12), the boundary $S^{n-1} = \{\tau = 0\}$ is non-characteristic. Hence, for any $h \in C^{k-1,\alpha}$, there exists a unique $C^{k-1,\alpha}$ smooth solution v to

$$D_2\Xi_{(g,f)}(v) = h$$

with initial value $v_0 = 0$ on S^{n-1}. This gives the existence of an inverse operator $L_{(g,f)}$ to $D_2 \Xi_{(g,f)}$, with a loss of 1-derivative. One has $L : Z_s \to Y_{s-1}$ with $D_2 \Xi_{(g,f)} \circ L = id$. The remaining conditions of (H4) are easily checked to hold. It follows then from [34, Cor. 2, p. 241] that for any g close to g_0 in $X_{2+\varepsilon}$ there exists $f \in Y_1$ (depending continuously on g), which satisfies (3.11) (and similarly for higher s).

This shows that, for any $g \in \mathrm{Met}^{k,\alpha}(C)$ close to g_0, with $k \geq 3$, there exists $f \in C^{k,\alpha'}(\partial C)$, which solves (3.11). Pulling back as above gives, for any initial $g \in C^{k,\alpha}$, a $C^{k-1,\alpha'}$ metric $\phi^* g$ in H-harmonic coordinates and satisfying (3.8). ∎

For the remainder of the proof, we work in the fixed H-harmonic coordinate system satisfying (3.8). Next, we derive the form of the Einstein equations for the metric g in (3.3). First, the second fundamental form $A = \frac{1}{2}\mathcal{L}_N g_{S_\tau}$ of the leaves S_τ has the form

$$A = \frac{1}{2}u^{-1}(\mathcal{L}_{\partial_\tau} g_S - \mathcal{L}_\sigma g_S), \tag{3.13}$$

where we have denoted by g_S the restriction of g on S_τ. More precisely, and since we shall compute on the $(n+1)$-dimensional manifold with tensors living on the n-dimensional slices S_τ,

$$g_S = g(\Pi_S \cdot, \Pi_S \cdot),$$

where Π_S is the orthogonal projection operator on S_τ. Thus, $g_S = g_{ij}(\mathrm{d}x^i + \sigma^i \mathrm{d}\tau)(\mathrm{d}x^j + \sigma^j \mathrm{d}\tau)$, as in (3.3). Clearly (3.13) is the same as

$$\mathcal{L}_{\partial_\tau} g_S = 2uA + \mathcal{L}_\sigma g_S. \tag{3.14}$$

A straightforward computation from commuting derivatives gives the Riccati equation

$$(\mathcal{L}_N A) = A^2 - u^{-1}(D^2 u) - R_N, \tag{3.15}$$

where $R_N = g_S(R(\cdot, N)N, \cdot)_{|TS \otimes TS}$ and A^2 is the bilinear form associated through g_S to the square of the shape operator of S_τ. (The Eq. (3.15) may also be derived from the second variation formula). Using the fact that A is tangential (i.e. $A(N, \cdot) = 0$), this gives

$$\partial_\tau A = -\mathcal{L}_\sigma A - D^2 u + uA^2 - uR_N. \tag{3.16}$$

Another straightforward calculation via the Gauss equations shows that $R_N = \mathrm{Ric}_g - \mathrm{Ric}_{S_\tau} + HA - A^2$, which, via (3.14) and (3.16) gives the system of 'evolution' equations for g_{ij} and $A = A_{ij}$ on S_τ:

$$\partial_\tau g = 2uA + \mathcal{L}_\sigma g_S, \tag{3.17}$$

$$\partial_\tau A = \mathcal{L}_\sigma A - D_S^2 u + u\left(\mathrm{Ric}_S - \mathrm{Ric}_g + 2A^2 - HA\right). \tag{3.18}$$

(Up to sign differences, these are the well-known Einstein evolution equations in general relativity, cf. [8,32].) Substituting the expression of A given by (3.17) in (3.18) gives the second-order evolution equation for g:

$$(\mathcal{L}_{\partial_\tau}\mathcal{L}_{\partial_\tau} + \mathcal{L}_\sigma\mathcal{L}_\sigma - 2\mathcal{L}_{\partial_\tau}\mathcal{L}_\sigma)g_S = u\mathrm{d}u(N)A - 2uD_S^2 u + 2u^2\left(\mathrm{Ric}_S - \mathrm{Ric}_g + 2A^2 - HA\right). \tag{3.19}$$

We now shift from these intrinsic equations to their expressions in coordinates. Any tangential 1-form on S_τ necessarily is of the form

$$\alpha = \alpha_i(\mathrm{d}x^i + \sigma^i \mathrm{d}\tau),$$

thus it is enough to work with the (i, j) components only. Using (2.7) (along the slices S_τ), one obtains

$$u^2 \Delta_S g_{ij} + \left((\mathcal{L}_{\partial_\tau}\mathcal{L}_{\partial_\tau} + \mathcal{L}_\sigma\mathcal{L}_\sigma - 2\mathcal{L}_{\partial_\tau}\mathcal{L}_\sigma)g_S\right)_{ij} = -2u^2(\mathrm{Ric}_g)_{ij} - 2u(D_S^2 u)_{ij} + Q_{ij}(g, \partial g), \tag{3.20}$$

where Q_{ij} is a term involving at most the first-order derivatives of $g_{\alpha\beta}$ in all x^α directions. Now,

$$(\mathcal{L}_{\partial_\tau} g_S)_{ij} = \partial_\tau g_{ij}, \qquad (\mathcal{L}_\sigma g_S)_{ij} = \sigma^k \partial_k g_{ij} + g_{kj}\partial_i \sigma^k + g_{ik}\partial_j \sigma^k,$$

so that, for Einstein metrics,

$$(\partial_\tau^2 + u^2 \Delta - 2\sigma^k \partial_k \partial_\tau + \sigma^k \sigma^l \partial_{kl}^2) g_{ij} = -2u(D^2 u)_{ij} + S_{ij}(g, \partial g) + Q_{ij}(g, \partial g), \tag{3.21}$$

where Q_{ij} has the same general form as before and S_{ij} contains tangential first and second derivatives of σ.

The $0i$ and 00 components of the Ricci curvature in the bulk are given by the 'constraint' equations along each leaf S_τ:

$$\begin{aligned} \delta(A - Hg) &= -\operatorname{Ric}_g(N, \cdot) = 0, \\ |A|^2 - H^2 + R_{S_\tau} &= R_g - 2\operatorname{Ric}_g(N, N) = (n-1)\lambda. \end{aligned} \tag{3.22}$$

Next, we derive the equations for the lapse u and shift σ along the leaves S_τ.

Lemma 3.4. *The lapse u and shift σ satisfy the following equations:*

$$\Delta u + |A|^2 u + \lambda u = -uN(H) = -(\partial_\tau - \sigma)H. \tag{3.23}$$

$$\Delta \sigma^i = -2u \langle D^2 x^i, A \rangle - u \langle dx^i, dH \rangle - 2 \left\langle dx^i, A(\nabla u) - \frac{1}{2} H du \right\rangle. \tag{3.24}$$

Proof. The lapse equation is derived by taking the trace of (3.15), and noting that

$$\operatorname{tr} \mathcal{L}_N A = N(H) + 2|A|^2.$$

For the shift equation, since the functions x^i are harmonic on S_τ, one has

$$\Delta((x^i)') + (\Delta')(x^i) = 0,$$

where $'$ denotes the Lie derivative with respect to uN and the Laplacian is taken with respect to the induced metric on the slices S_τ. Moreover $(x^i)' = -\sigma^i$ (see above), and from standard formulas, cf. [11, Ch. 1K] for example, one has

$$(\Delta')(x^i) = -2\langle D^2 x^i, \delta^* uN \rangle + 2\langle dx^i, \beta(\delta^* uN) \rangle,$$

where all the terms on the right are along S_τ and β is the Bianchi operator, $\beta(k) = \delta k + \frac{1}{2} d\operatorname{tr} k$. Thus, $\delta^*(uN) = uA$, and the shift components σ^i satisfy

$$\Delta \sigma^i = -2u \langle D^2 x^i, A \rangle + 2u \left\langle dx^i, \delta A - \frac{1}{2} dH \right\rangle + 2 \left\langle dx^i, A(\nabla u) - \frac{1}{2} H du \right\rangle.$$

The relation (3.24) then follows from the constraint equation (3.22). ∎

Summarizing the work above, the Einstein equations in local H-harmonic coordinates imply the following system on the data (g_{ij}, u, σ):

$$(\partial_\tau^2 + u^2 \Delta - 2\sigma^k \partial_k \partial_\tau + \sigma^k \sigma^l \partial_{kl}^2) g_{ij} = -2u(D^2 u)_{ij} + S_{ij}(g_{\alpha\beta}, \partial g_{\alpha\beta}) + Q_{ij}(g_{\alpha\beta}, \partial g_{\alpha\beta}), \tag{3.25}$$

$$\Delta u + |A|^2 u + \lambda u = dH_0(\sigma). \tag{3.26}$$

$$\Delta \sigma^i = -2u \langle D^2 x^i, A \rangle - u \partial_i H_0 - 2 \left(A_j^i \nabla^j u - \frac{1}{2} H_0 \nabla^i u \right), \tag{3.27}$$

where H_0 denotes the mean curvature of the $\{\tau = 0\}$-slice U.

Remark 3.5. The system (3.25)–(3.27) is essentially an elliptic system in (g_{ij}, u, σ), given that $H = H_0$ is prescribed. Thus, assuming $u \sim 1$ and $\sigma \sim 0$, the operator $P = \partial_\tau^2 + u^2 \Delta - 2\sigma^i \partial_i \partial_\tau + \sigma^k \sigma^l \partial_{kl}^2$ is elliptic on C and acts diagonally on $\{g_{ij}\}$, as is the Laplace operator on the slices S_τ acting on (u, σ). The system (3.25)–(3.27) is of course coupled, but the couplings are all of lower order, i.e. first order, except for the term $D^2 u$ in (3.25). However, this term can be controlled or estimated by elliptic regularity applied to the lapse equation (3.26) (as discussed further below). Given the above, it is not difficult to deduce that local H-harmonic coordinates have the optimal regularity property, i.e. if g is in $C^{m,\alpha}(C)$ in some local coordinate system, then g is in $C^{m,\alpha}(C)$ in H-harmonic coordinates. Since this will not actually be used here, we omit further details of the proof.

Next we show that the lapse and shift, and their τ-derivatives, are determined by the tangential metric g_S and its τ-derivative.

Lemma 3.6. *Suppose the metric g is close to the Euclidean metric in the $C^{2,\alpha}$ topology. Then in local H-harmonic coordinates (τ, x^i) as defined above, the lapse–shift components (u, σ^i) and their derivatives $(\partial_\tau u, \partial_\tau \sigma^i)$, are uniquely determined either by the tangential metric g_{ij} and 2nd fundamental form A_{ij} on each S_τ, or by the tangential metric g_{ij} and its time derivatives $\partial_\tau g_{ij}$ on each S_τ.*

Proof. The system (3.26) and (3.27) is a coupled elliptic system in the pair (u, σ) on S_τ, with boundary values on ∂S_τ given by

$$u|_{\partial S_\tau} = 1, \qquad \sigma|_{\partial S_\tau} = 0. \tag{3.28}$$

In the x^i coordinates, all the coefficients of (3.26) and (3.27) are bounded in C^α. Since the metric g_{ij} is close to the flat metric in the $C^{2,\alpha}$ topology, it is standard that there is then a unique solution to the elliptic boundary value problem (3.26)–(3.28), cf. [19]. The solution (u, σ) is uniquely determined by the coefficients (g_{ij}, A_{ij}) and the terms or coefficients containing derivatives of H and the x^i. But these are also determined by (g_{ij}, A_{ij}). Combining the facts above, it follows that (u, σ) is uniquely determined by (g_{ij}, A_{ij}).

The second claim is obtained in the same manner: rewrite the equations by replacing all the occurrences of A_{ij} by its expression in (3.13). The equations are then non-linear equations in (u, σ^i). Considering them as a non-linear operator from $C^{2,\alpha}$ to C^α depending also on the metric, a simple computation shows that the operator linearized at the Euclidean metric is invertible. Invertibility of the non-linear operator then follows from the implicit function theorem.

Next we claim that $\partial_\tau g_{0\alpha}$ is also determined by (g_{ij}, A_{ij}) along S_τ. To see this, first note that

$$\mathcal{L}_N g = \mathcal{L}_N g_S + \mathcal{L}_N(g(N, \cdot)) \otimes g(N, \cdot) + g(N, \cdot) \otimes \mathcal{L}_N(g(N, \cdot)),$$

where $\mathcal{L}_N g_S = 2A$, $g(N, \cdot) = u\mathrm{d}\tau$ and

$$\mathcal{L}_N(g(N, \cdot))(\partial_i) = -g(N, [u^{-1}(\partial_\tau - \sigma, \partial_i]) = u^{-1}\mathrm{d}u(\partial_i), \quad \mathcal{L}_N(g(N, \cdot)))(N) = 0.$$

This shows that all components of $\mathcal{L}_N g$ are determined by (g_{ij}, A_{ij}) (since u and σ are already so determined). Now write $\mathcal{L}_N g(\partial_\tau, \partial_\alpha) = N(g_{0\alpha}) + l_{0\alpha}$. One has $N(g_{0\alpha}) = u^{-1}\partial_\tau g_{0\alpha} - u^{-1}\partial_\sigma g_{0\alpha}$, and the second term is again determined by (g_{ij}, A_{ij}). Calculating the term l_{0i} above explicitly, one easily finds that it also depends only on (g_{ij}, A_{ij}), so that

$$\partial_\tau g_{0i} = \phi_i,$$

is determined by an explicit formula in $g_{ij}, A_{ij}, u, \sigma$ and their tangential derivatives, and so implicitly by g_{ij}, A_{ij}. Working now in the same way shows that the same is true for $\partial_\tau g_{00}$. This completes the proof. ∎

Proof of Theorems 3.1 and 1.1. Suppose that g and $\widetilde{g}$ are two Einstein metrics on C with identical (γ, A) on $\partial_0 C = U$. One may construct H-harmonic coordinates for each, and via a diffeomorphism identifying these coordinates, assume that the resulting pair of metrics g and $\widetilde{g}$ have fixed H-harmonic coordinates (τ, x^i), and both metrics satisfy the system (3.25)–(3.27). Let

$$h = h_{ij} = \widetilde{g}_{ij} - g_{ij}. \tag{3.29}$$

One then takes the difference of both Eqs. (3.25) and freezes the coefficients at g to obtain a linear equation in h. Thus, for example, $\Delta_{\widetilde{g}}\widetilde{g}_{ij} - \Delta_g g_{ij} = \Delta_g(h_{ij}) - (g^{ab} - \widetilde{g}^{ab})\partial_a\partial_b\widetilde{g}_{ij}$. The second term here is of zero order (rational), in the difference h, with coefficients depending on two derivatives of $\widetilde{g}$. Carrying out the same procedure on the remaining terms in (3.25) gives the equation

$$(\partial_\tau^2 + u^2\Delta - 2\partial_\sigma\partial_\tau + \partial_\sigma^2)h_{ij} = -2(\widetilde{u}(\widetilde{D}^2\widetilde{u})_{ij} - u(D^2u)_{ij}) + \mathcal{Q}_{ij}(h_{\alpha\beta}, \partial_\mu h_{\alpha\beta}),$$

where we have denoted $\partial_\sigma = \sigma^k\partial_k$, $\partial_\sigma^2 = \sigma^k\sigma^l\partial_{kl}$, and $\mathcal{Q}$ is a term depending on two derivatives of the background $(g, \widetilde{g})$ and linear in its arguments, whose precise value may change from line to line. Similarly, $\widetilde{D}^2\widetilde{u} - D^2u = D^2v + (\widetilde{D}^2 - D^2)\widetilde{u}$, where $v = \widetilde{u} - u$ and the second term is of the form Q above. Hence,

$$(\partial_\tau^2 + u^2\Delta - 2\partial_\sigma\partial_\tau + \partial_\sigma^2)h_{ij} = -2u(D^2v)_{ij} + \mathcal{Q}_{ij}(h_{\alpha\beta}, \partial_\mu h_{\alpha\beta}). \tag{3.30}$$

Note that since we have linearized, $\mathcal{Q}$ depends linearly on $h_{\alpha\beta}$ and $\partial_\mu h_{\alpha\beta}$, with non-linear coefficients depending on $\widetilde{g}$ and g.

Next we use the lapse and shift equations (3.26) and (3.27) to estimate the differences $v = \widetilde{u} - u$ and $\chi = \widetilde{\sigma} - \sigma$. Thus, as before, $\Delta_{\widetilde{g}} \widetilde{u} - \Delta_g u = \Delta_g v + D_h^2(\widetilde{u})$, where D_h^2 is a second-order differential operator on $\widetilde{u}$ with coefficients depending on the difference h, to first order. The remaining terms in (3.26) and (3.27) can all be treated in the same way, using (3.13) to replace occurrences of A_{ij} by derivatives in τ and σ. Taking the difference, it then follows from (3.26) and (3.27) that

$$\begin{cases} \Delta v + |A|^2 v + \lambda v = Q(h_{ij}, \partial_\mu h_{ij}) \\ \Delta \chi^i + 2v \langle D^2 x^i, A \rangle + v \partial_i H_0 + 2 \left(A_j^i - \frac{1}{2} H \delta_j^i \right) g^{jk} \partial_k u = Q^i(h_{kl}, \partial_\mu h_{kl}), \end{cases} \tag{3.31}$$

where the terms Q are linear in the arguments and their coefficients depend on one derivatives of $\widetilde{g}$. Note also that the zeroth- and first-order terms in (v, χ) are small if the metric is close to the Euclidean metric. Thus, the left-hand side operators are invertible with $v = 0$ and $\chi = 0$ on ∂S_τ, and elliptic regularity applied to the system (3.31) then gives

$$\|v\|_{L_x^{2,2}} \leq C(\widetilde{g}, g) \|h_{ij}\|_{L_{(\tau,x)}^{1,2}}, \tag{3.32}$$

and

$$\|\chi\|_{L_x^{2,2}} \leq C(\widetilde{g}, g) \|h_{ij}\|_{L_{(\tau,x)}^{1,2}}. \tag{3.33}$$

It follows from (3.30), (3.32) and (3.33) that

$$\|P(h_{ij})\|_{L_x^2} \leq C(\widetilde{g}_{\alpha\beta}, g_{\alpha\beta}) \|h_{\alpha\beta}\|_{L_{(\tau,x)}^{1,2}}, \tag{3.34}$$

where P is given in Remark 3.5.

Now by applying Lemma 3.6 to (u, σ) and $(\widetilde{u}, \widetilde{\sigma})$ and taking the difference as above, it follows that v and χ, as well as $\partial_\tau v$ and $\partial_\tau \chi$ are given by a linear expression in h_{ij} and its first derivatives (in every direction). Hence, (3.34) becomes

$$\|P(h_{ij})\|_{L_x^2} \leq C(\widetilde{g}, g) \|h_{ij}\|_{L_{(\tau,x)}^{1,2}}. \tag{3.35}$$

We are now in position to apply the Calderón unique continuation theorem [14]. Thus, the operator P is elliptic and diagonal, and the Cauchy data for P vanish at U, i.e.

$$h = \partial_\tau h = 0 \quad \text{at } U. \tag{3.36}$$

We claim that P satisfies the hypotheses of the Calderon unique continuation theorem [14]. Following [14], decompose the symbol of P as

$$\begin{aligned} A_2(\tau, x, \xi) &= (u^2 g^{kl} \xi_k \xi_l - 2\sigma^k \sigma^l \xi_k \xi_l) I, \\ A_1(\tau, x, \xi) &= \sigma^k \xi_k I, \end{aligned} \tag{3.37}$$

where I is the $N \times N$ identity matrix, $N = \frac{1}{2} n(n+1)$, equal to the cardinality of $\{ij\}$. Setting $|\xi|^2 = 1$, (3.37) becomes

$$\begin{aligned} A_2(\tau, x, \xi) &= (u^2 - 2\sigma^k \sigma^l \xi_k \xi_l) I, \\ A_1(\tau, x, \xi) &= \sigma^k \xi_k I. \end{aligned}$$

Now form the matrix

$$M = \begin{pmatrix} 0 & -I \\ A_2 & A_1 \end{pmatrix}. \tag{3.38}$$

The matrices A_1 and A_2 are diagonal, and it is then easy to see that M is diagonalizable, i.e. has a basis of eigenvectors over $\mathbb{C}$. This implies that M satisfies the hypotheses of [14, Thm. 11(iii)], cf. also [14, Thm. 4]. The bound (3.35) is substituted in the basic Carleman estimate of [14, Thm. 6], cf. also [29, (6.1)], showing that h_{ij} satisfies the unique continuation property. It follows from (3.36) and the Calderón unique continuation theorem that

$$h_{ij} = \widetilde{g}_{ij} - g_{ij} = 0,$$

in an open neighborhood $\Omega \subset C$.

By Lemma 3.6 once again, this implies $h_{\alpha\beta} = 0$, i.e.

$$\widetilde{g}_{\alpha\beta} = g_{\alpha\beta},$$

in Ω, so that $\widetilde{g}$ is isometric to g in Ω. By construction, the isometry from $\widetilde{g}$ to g equals the identity on U. This shows that the metric g is uniquely determined in Ω, up to isometry, by the abstract Cauchy data on U. Since Einstein metrics are real–analytic in the interior in harmonic coordinates, a standard analytic continuation argument (cf. [25] for instance), then implies that g is unique up to isometry everywhere in C. This completes the proof of Theorem 3.1.

In the context of Theorem 1.1, the same analytic continuation argument shows that a pair of Einstein metrics (M_i, g_i), $i = 0, 1$, whose Cauchy data agree on a common open set U of ∂M_i are everywhere locally isometric, i.e. they become isometric in suitable covering spaces, modulo restriction or extension of the domain, as discussed following Theorem 1.1. This then also completes the proof of Theorem 1.1. ∎

As an illustration, suppose (M_1, g_1) and (M_2, g_2) are a pair of Einstein metrics on compact manifolds with boundary and the Cauchy data for g_1 and g_2 agree on an open set U of the boundary. Suppose M_i are connected and the topological condition (1.5) holds for each M_i. Then, modulo isometry, either $M_1 \subset M_2$, $M_2 \subset M_1$, or M_i are subdomains in a larger Einstein manifold $M_3 = M_1 \cup M_2$.

We conclude this section with a discussion of generalizations of Theorem 1.1. First, one might consider the unique continuation problem for

$$\text{Ric}_g = T, \tag{3.39}$$

where T is a fixed symmetric bilinear form on M, at least C^α up to $\bar{M}$. However, this problem is not natural, in that is not covariant under changes by diffeomorphism. For metrics alone, the Einstein equation (1.2) is the only equation covariant under diffeomorphisms which involves at most the second derivatives of the metric. Nevertheless, the proof of Theorem 1.1 shows that if $\widetilde{g}$ and g are two solutions of (3.39) which have common H-harmonic coordinates near (a portion of) ∂M on which $(\gamma, A) = (\widetilde{\gamma}, \widetilde{A})$, then $\widetilde{g}$ is isometric to g near (a portion of) ∂M.

Instead, it is more natural to consider the Einstein equation coupled (covariantly) to other fields χ besides the metric; such equations arise naturally in many areas of physics. For example, χ may be a function on M, i.e. a scalar field, or χ may be a connection 1-form (gauge field) on a bundle over M. We assume that the field(s) χ arise via a diffeomorphism-invariant Lagrangian $\mathcal{L} = \mathcal{L}(g, \chi)$, depending on χ and its first derivatives in local coordinates, and that χ satisfies field equations, i.e. Euler–Lagrange equations, coupled to the metric. For example, for a free massive scalar field, the equation is the eigenfunction equation

$$\Delta_g \chi = \mu \chi, \tag{3.40}$$

while for a connection 1-form, the equations are the Yang–Mills equations (or Maxwell equations when the bundle is a $U(1)$ bundle):

$$\mathrm{d}F = \mathrm{d}^* F = 0, \tag{3.41}$$

where F is the curvature of the connection χ. Associated to such fields is the stress-energy tensor $T = T_{\mu\nu}$; this is a symmetric bilinear form obtained by varying the Lagrangian for χ with respect to the metric, cf. [22] for example. For the free massive scalar field χ above, one has

$$T = \mathrm{d}\chi \cdot \mathrm{d}\chi - \frac{1}{2}(|\mathrm{d}\chi|^2 + \mu\chi^2)g,$$

while for a connection 1-form

$$T = F \cdot F - \frac{1}{4}|F|^2 g,$$

where $(F \cdot F)_{ab} = F_{ac}F_{bd}g^{cd}$.

When the part of the Lagrangian involving the metric to second order only contains the scalar curvature, i.e. the Einstein–Hilbert action, the resulting coupled Euler–Lagrange equations for the system (g, χ) are

$$\text{Ric}_g - \frac{R}{2}g = T, \qquad E_g(\chi) = 0. \tag{3.42}$$

By taking the trace, this can be rewritten as

$$\text{Ric}_g = \hat{T} = T - \frac{1}{n-1} tr_g T, \qquad E_g(\chi) = 0. \tag{3.43}$$

Here we assume $E_g(\chi)$ is a second-order elliptic system for χ, with coefficients depending on g, as in (3.40) or (3.41) (the latter viewed as an equation for the connection). In case the field(s) χ have an internal symmetry group, as in the case of gauge fields, this will require a particular choice of gauge for χ in which the Euler–Lagrange equations become an elliptic system in χ. It is also assumed that solutions χ of $E_g(\chi) = 0$ satisfy the unique continuation property; for instance E_g satisfies the hypotheses of the Calderón theorem [14]. Theorem 1.1 now easily extends to cover (3.42) or (3.43).

Proposition 3.7. *Let M be a compact manifold with boundary ∂M. Then $C^{3,\alpha}$ solutions (g, χ) of (3.42) on $\bar{M}$ are uniquely determined, up to local isometry and inclusion, by the Cauchy data (γ, A) of g and the Cauchy data $(\chi, \partial_t \chi)$ on an open set $U \subset \partial M$.*

Proof. The proof is the same as the proof of Theorem 1.1. Briefly, via a suitable diffeomorphism equal to the identity on ∂M, one brings a pair of solutions of (3.42) with common Cauchy data into a fixed system of H-harmonic coordinates for each metric. As before, one then applies Calderón uniqueness to the resulting system (3.42) in the difference of the metrics and fields. Further details are left to the reader. ∎

4. Proof of Theorem 1.2

Let g be a conformally compact metric on a compact $(n + 1)$-manifold M with boundary which has a C^2 geodesic compactification

$$\bar{g} = t^2 g, \tag{4.1}$$

where $t(x) = \text{dist}_{\bar{g}}(x, \partial M)$. By the Gauss lemma, one has the splitting

$$\bar{g} = dt^2 + g_t, \tag{4.2}$$

near ∂M, where g_t is a curve of metrics on ∂M with $g_0 = \gamma$ the boundary metric. The curve g_t is obtained by taking the induced metric the level sets $S(t)$ of t, and pulling back by the flow of $N = \bar{\nabla} t$. Note that if $r = -\log t$, then $g = dr^2 + t^{-2} g_t$, so the integral curves of ∇r with respect to g are also geodesics. Each choice of boundary metric $\gamma \in [\gamma]$ determines a unique geodesic defining function t.

Now suppose g is Einstein, so that (1.4) holds and suppose for the moment that g is C^2 conformally compact with C^∞ smooth boundary metric γ. Then the boundary regularity result of [16] implies that $\bar{g}$ is C^∞ smooth when n is odd, and is C^∞ polyhomogeneous when n is even. Hence, the curve g_t has a Taylor-type series in t, called the Fefferman–Graham expansion [18]. The exact form of the expansion depends on whether n is odd or even. If n is odd, one has a power series expansion

$$g_t \sim g_{(0)} + t^2 g_{(2)} + \cdots + t^{n-1} g_{(n-1)} + t^n g_{(n)} + \cdots, \tag{4.3}$$

while if n is even, the series is polyhomogeneous,

$$g_t \sim g_{(0)} + t^2 g_{(2)} + \cdots + t^n g_{(n)} + t^n \log t \, \mathcal{H} + \cdots. \tag{4.4}$$

In both cases, this expansion is even in powers of t, up to t^n. It is important to observe that the coefficients $g_{(2k)}$, $k \leq [n/2]$, as well as the coefficient $\mathcal{H}$ when n is even, are explicitly determined by the boundary metric $\gamma = g_{(0)}$ and the Einstein condition (1.4), cf. [18,20]. For n even, the series (4.4) has terms of the form $t^{n+k}(\log t)^m$.

For any n, the divergence and trace (with respect to $g_{(0)} = \gamma$) of $g_{(n)}$ are determined by the boundary metric γ; in fact there is a symmetric bilinear form $r_{(n)}$ and scalar function $a_{(n)}$, both depending only on γ and its derivatives up to order n, such that

$$\delta_\gamma(g_{(n)} + r_{(n)}) = 0, \quad \text{and} \quad tr_\gamma(g_{(n)} + r_{(n)}) = a_{(n)}. \tag{4.5}$$

For n odd, $r_{(n)} = a_{(n)} = 0$. (The divergence-free tensor $g_{(n)} + r_{(n)}$ is closely related to the stress energy of a conformal field theory on $(\partial M, \gamma)$, cf. [17]). The relations (4.5) will be discussed further in Section 5.

However, beyond the relations (4.5), the term $g_{(n)}$ is not determined by $g_{(0)}$; it depends on the "global" structure of the metric g. The higher order coefficients $g_{(k)}$ of t^k and coefficients $h_{(km)}$ of $t^{n+k}(\log t)^m$, are then determined by $g_{(0)}$ and $g_{(n)}$ via the Einstein equations. The Eqs. (4.5) are constraint equations, and arise from the Gauss–Codazzi and Gauss and Riccati equations on the level sets $S(t) = \{x : t(x) = t\}$ in the limit $t \to 0$; this is also discussed further in Section 5.

In analogy to the situation in Section 3, the term $g_{(n)}$ corresponds to the second fundamental form A of the boundary, in that, modulo the constraints (4.5), it is freely specifiable as Cauchy data, and is the only such term depending on normal derivatives of the boundary metric.

Suppose now g_0 and g_1 are two solutions of

$$\text{Ric}_g + ng = 0, \tag{4.6}$$

with the same C^∞ conformal infinity $[\gamma]$. Then there exist geodesic defining functions t_k such that $\bar{g}_k = (t_k)^2 g_k$ have a common boundary metric $\gamma \in [\gamma]$, and both metrics are defined for $t_k \leq \varepsilon$, for some $\varepsilon > 0$.

The hypotheses of Theorem 1.2, together with the discussion above concerning (4.3) and (4.4), then imply that

$$|g_1 - g_0| = o(e^{-nr}) = o(t^n), \tag{4.7}$$

where the norm is taken with respect to g_1 (or g_0).

Given this background, we prove the following more general version of Theorem 1.2, analogous to Theorem 3.1. Let Ω be a domain diffeomorphic to $I \times B^n$, where B^n is a ball in $\mathbb{R}^n$ with boundary $U = \partial\Omega$ diffeomorphic to a ball in $\mathbb{R}^n \simeq \{0\} \times \mathbb{R}^n$.

Theorem 4.1. *Let g_0 and g_1 be a pair of conformally compact Einstein metrics on a domain Ω as above. Suppose g_0 and g_1 have $C^{3,\alpha}$ geodesic compactifications, and (4.7) holds in Ω.*

Then (Ω, g_0) is isometric to (Ω, g_1), by an isometry equal to the identity on $\partial\Omega$. Hence, if (M_0, g_0) and (M_1, g_1) are conformally compact Einstein metrics on compact manifolds with boundary, and (4.7) holds on some open domain Ω in M_0 and M_1, then the manifolds M_0 and M_1 are diffeomorphic in some covering space of each and the lifted metrics g_0 and g_1 are isometric.

The proof of Theorem 4.1 is very similar to that of Theorem 3.1. For clarity, we first prove the result in case the metrics g_i, $i = 0, 1$, have a common C^∞ boundary metric γ and then show how the proof can be extended to cover the more general case of metrics with less regularity.

By applying a diffeomorphism if necessary, one may assume that the metrics g_i have a common geodesic defining function t defined near $\partial\Omega$ and common geodesic boundary coordinates. By [16], the geodesically compactified metrics $\bar{g}_i = t^2 g_i$ are C^∞ polyhomogeneous and extend C^∞ polyhomogeneously to $\partial\Omega$. It follows from the discussion of the Fefferman–Graham expansion following (4.5) that g_0 and g_1 agree to infinite order at ∂U, i.e.

$$k = g_1 - g_0 = O(t^\nu), \tag{4.8}$$

for any $\nu < \infty$. Of course $k_{0\alpha} = 0$.

For the rest of the proof, we work in the setting of the compactified metrics $\bar{g}_i$. As in the proof of Theorem 3.1, we assume that the domain Ω, now denoted C, is sufficiently small so that $(C, \bar{g}_i)$ is close to the flat metric on the standard cylinder $C = I \times B^n$, with $\bar{A} = 0$ on $U = \partial_0 C$. (Note that $g_{(1)} = 0$ in (4.3), (4.4)). In particular, near $\partial_0 C$, $\bar{H} = O(t)$. One may construct a foliation S_τ with $\bar{H}_{S_\tau} = 0$, together with corresponding H-harmonic coordinates (τ, x^i), exactly as in Lemmas 3.2 and 3.3, and satisfying the boundary conditions (3.8). All of the analysis carried out in Section 3 carries over to this situation with only a single difference. Namely, for the term Ric_g in (3.19) or (3.20), one now no longer has $\text{Ric}_g = \lambda g$, but instead the Ricci curvature $\bar{\text{Ric}}$ of the compactified metric $\bar{g}$. Using the facts that $\text{Ric}_g = -ng$ and the compactification $\bar{g}$ is geodesic, standard formulas for the behavior of Ricci curvature under conformal change give

$$\bar{\text{Ric}} = -(n-1)t^{-1}\bar{D}^2 t - t^{-1}\bar{\Delta}t\bar{g}. \tag{4.9}$$

One has $\bar{D}^2 t = \mathcal{L}_{\nabla t} \bar{g} = O(t)$. If (t, y^i) are geodesic boundary coordinates, then $\partial_{x^i} = \sum (1 - \varepsilon(\tau))\partial_{y^j} + \varepsilon(\tau)\nabla t$, where $\varepsilon(\tau) = O(\tau)$. Similarly, $\tau/t = 1 + \varepsilon(\tau)$. (The specific form of $\varepsilon(\tau)$ of course differs in each occurrence above, but this is insignificant). Since $\bar{D}^2 t$ vanishes on ∇t, it follows from (4.9) that in the x^i coordinates on S_τ,

$$\bar{\mathrm{Ric}}_{ij} = -(n-1)(1-\varepsilon)^2 t^{-1}(\mathcal{L}_{\nabla t}\bar{g})_{ij} - (1-\varepsilon)^2 t^{-1}(\bar{\Delta}t)\bar{g}_{ij} + \varepsilon t^{-1}(\bar{\Delta}t)q_{ij}, \tag{4.10}$$

where q_{ij} depends only on $\bar{g}_{0\alpha}$ to zero order. Next $(\mathcal{L}_{\nabla t}\bar{g}) = (1-\varepsilon)\partial_\tau \bar{g} + \varepsilon(\tau)\partial_{x^\alpha}\bar{g}$ and similarly for the Laplace term in (4.10). Substituting (4.10) in (3.20), it follows that the analog of (3.21) in this context is the 'evolution equation'

$$\tau^2(\partial_\tau^2 + u^2\Delta - 2\partial_\sigma\partial_\tau + \partial_\sigma^2)g_{ij} = -2\tau^2 u(D^2 u)_{ij} + S_{ij}(g, \tau\partial g) + Q_{ij}(g, \tau\partial g), \tag{4.11}$$

where S_{ij} and Q_{ij} have the same meaning as before. Here and below, we drop the bar from the notation.

The lapse u and shift σ satisfy essentially the same equations as before, namely

$$\Delta u + |A|^2 u - (t^{-1}\Delta t)u = 0, \tag{4.12}$$

$$\Delta\sigma^i = -2u\langle D^2 x^i, A\rangle - 2\langle dx^i, A(\nabla u)\rangle. \tag{4.13}$$

Comparing with (3.23), (3.24), one has here $H = 0$, with the λ term in replaced by $-t^{-1}\Delta t$. Lemma 3.6 holds as before, since $t^{-1}\Delta t$ is smooth up to $\partial_0 C$.

One now proceeds just as in the proof of Theorem 3.1, taking the difference of the Eq. (4.11) to obtain a linear equation on $h = \widetilde{g} - g$; (recall that the bars have been removed from the notation). Note that by (4.8), together with elliptic regularity applied to (4.12) and (4.13), as in the proof of Lemma 3.6, one has

$$h_{\alpha\beta} = O(t^\nu), \tag{4.14}$$

for all $\nu < \infty$. The estimates (3.32)–(3.35) hold as before.

Let $P(h_{ij}) = \tau^2(\partial_\tau^2 + u^2\Delta - 2\partial_\sigma\partial_\tau + \partial_\sigma^2)$. Then P is a fully degenerate second-order elliptic operator, with smooth coefficients, and one has

$$\|P(h_{ij})\|_{L_x^2} \le C\|h_{ij}\|_{L_{\tau,x}^{1,2}},$$

where the first-order derivatives on the right are of the form $\tau\partial$. Further, by (4.14), h vanishes to infinite order at $\partial_0 C$. It then follows from a unique continuation theorem of Mazzeo, [27, Thm. 14], that

$$h_{ij} = 0$$

in $\Omega \subset C$. The vanishing of $h = h_{\alpha\beta}$ in C then follows as before in the proof of Theorem 3.1.

Next suppose g_0 and g_1 have only a $C^{3,\alpha}$ geodesic compactification with a common boundary metric γ, but that (4.7) holds. All of the arguments above remain valid, except the infinite order vanishing property (4.8), and the corresponding (4.14), which are replaced by the statements $k = o(t^n)$ and $h = o(t^n)$ respectively. The unique continuation result in [27] per se, requires the infinite order decay (4.14). Thus, it suffices to show that (4.14) does in fact hold.

To do this, we first show that $k = O(t^\nu)$ weakly, for all $\nu < \infty$. This will imply $h = O(t^\nu)$ weakly, and the strong or pointwise decay (4.14) then follows from elliptic regularity.

In geodesic boundary coordinates, the geodesic compactification of a conformally compact Einstein metric satisfies the equation

$$t\ddot{g} - (n-1)\dot{g} - 2Hg^T - 2t\,\mathrm{Ric}_{S(t)} + tH\dot{g} - t(\dot{g})^2 = 0, \tag{4.15}$$

where $\dot{g}$ is the Lie derivative of g with respect to ∇t, cf. [18] or [21]. Thus $\dot{g} = 2A$, where A is the second fundamental form of the level set $S(t)$ of t (with respect to the inward normal). Also $H = trA$, T denotes restriction or projection onto $S(t)$ and $\mathrm{Ric}_{S(t)}$ is the intrinsic Ricci curvature of $S(t)$. (The Eq. (4.15) may be derived from (3.18) by setting $u = 1$ and $\sigma = 0$.) We recall, as above, that the bar has been removed from the notation.

As above, the metrics g_0 and g_1 are assumed to have a fixed geodesic defining function t with common boundary metric γ and common geodesic boundary coordinates. Taking the difference of the Eq. (4.15) evaluated on g_1 and g_0 gives the following equation for $k = g_1 - g_0$ as in (4.8):

$$t\ddot{k} - (n-1)\dot{k} = tr(\dot{k})g_0^T + 2t(\mathrm{Ric}_{S(t)}^1 - \mathrm{Ric}_{S(t)}^0) + O(t)k + O(t^2)\dot{k}, \tag{4.16}$$

where $O(t^k)$ denotes terms of order t^k with coefficients depending smoothly on g_0. One has $\mathrm{Ric}_{S(t)} = D_x^2(g_{ij})$ is a second-order operator on g_{ij}, so that (4.16) gives

$$t\ddot{k} - (n-1)\dot{k} = tr(\dot{k})g_0^T + 2t D_x^2(k) + O(t)k + O(t^2)\dot{k}. \tag{4.17}$$

The (positive) indicial root of the trace-free part of (4.16) or (4.17) is n, in that the formal power series solution of (4.17) has undetermined coefficient at order t^n, as in the Fefferman–Graham expansion (4.3) and (4.4). The hypothesis (4.7) implies that

$$k = o(t^n), \tag{4.18}$$

so that this nth-order coefficient vanishes. However, taking the trace of (4.17) gives

$$ttr\ddot{k} - (2n-1)ttr\dot{k} = tr(O(t)k + O(t^2\dot{k})) + 2ttr(D_x^2(k)),$$

which has indicial root $2n$. To see that trk is in fact formally determined at order $2n$, one uses the trace of the Riccati equation (3.15) (with $u = 1$ and $\sigma = 0$), which gives

$$\dot{H} + |A|^2 = -\mathrm{Ric}(T, T). \tag{4.19}$$

Via (4.9), this is easily seen to be equivalent to

$$t\dot{H} - H = -t|A|^2.$$

This holds for each compactified metric g_1 and g_0, and so taking the difference, and computing as in (4.16) and (4.17) gives the equation

$$t\frac{\mathrm{d}^2}{\mathrm{d}t^2}(trk) - \frac{\mathrm{d}}{\mathrm{d}t}(trk) = O(t)k + O(t^2)\dot{k}. \tag{4.20}$$

The positive indicial root of (4.20) is 2, and by (4.7), the $O(t^2)$ component of the formal expansion of trk vanishes. Similarly, the trace-free part k_0 of k satisfies the equation

$$t\ddot{k}_0 - (n-1)\dot{k}_0 = 2t(D_x^2(k))_0 + [O(t)k]_0 + [O(t^2)\dot{k}]_0, \tag{4.21}$$

with indicial root n. As in [18], by repeated differentiation of (4.20) and (4.21) it follows from (4.7) that the formal expansion of k vanishes.

Next we show that (4.8) holds weakly.

Lemma 4.2. *Suppose $k = o(t^n)$ weakly, in that, with respect to the compactified metric $(S(t), g)$, $(g = g_0)$,*

$$\int_{S(t)} \langle k, \phi \rangle = o(t^n), \quad as \ t \to 0, \tag{4.22}$$

where ϕ is any symmetric bilinear form, C^∞ smooth up to $U = \partial_0 C$ and vanishing to infinite order on ∂C. Then

$$k = o(t^\nu), \quad weakly, \tag{4.23}$$

for any $\nu < \infty$, i.e. (4.22) holds, with ν in place of n.

Proof. Here smoothness is measured with respect to the given geodesic coordinates (t, x^i) covering C. The proof proceeds by induction, starting at the initial level n. As above, the trace-free and pure trace cases are treated separately, and so we assume in the following first that ϕ is trace-free. Pair k with ϕ and integrate (4.17) over the level sets $S(t)$ to obtain

$$t\int_{S(t)} \langle \ddot{k}, \phi \rangle - (n-1)\int_{S(t)} \langle \dot{k}, \phi \rangle = t\int_{S(t)} \langle k, P_2(\phi) \rangle + \int_{S(t)} \langle O(t)k, \phi \rangle + \int_{S(t)} \langle O(t^2)\dot{k}, \phi \rangle. \tag{4.24}$$

Here $P_2(\phi)$ is obtained by integrating the D_x^2 term on the right in (4.17) by parts over $S(t)$. Thus $P_2(\phi)$, and more generally, $P_k(\phi)$ denote differential operators of order k on ϕ with coefficients depending on g and g_1 and their

derivatives up to order 2 and so at least continuous up to $\bar{\partial}\Omega$. We use these expressions generically, so their exact form may change from line-to-line below. Note also there are no boundary terms at $\partial S(t)$ arising from the integration by parts, by the vanishing hypothesis on ∂C.

For the terms on the right in (4.24) one then has

$$\int_{S(t)} \langle O(t)k, \phi \rangle = t \int_{S(t)} \langle k, P_0(\phi) \rangle,$$

while, since $A = O(t)$ and $H = O(t)$,

$$\int_{S(t)} \langle O(t^2)\dot{k}, \phi \rangle = t^2 \int_{S(t)} \langle \dot{k}, P_0(\phi) \rangle = t^2 \frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} \langle k, P_0(\phi) \rangle - t^2 \int_{S(t)} \langle k, P_1(\phi) \rangle.$$

Similarly, for the terms on the left in (4.24), one has

$$\int_{S(t)} \langle \dot{k}, \phi \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} \langle k, \phi \rangle - t \int_{S(t)} \langle k, P_1(\phi) \rangle,$$

while

$$\int_{S(t)} \langle \ddot{k}, \phi \rangle = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{S(t)} \langle k, \phi \rangle - 2t \frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} \langle k, P_1(\phi) \rangle + \int_{S(t)} \langle k, P_1(\phi) \rangle + t \int_{S(t)} \langle k, P_2(\phi) \rangle.$$

Now let

$$f = f(t) = \int_{S(t)} \langle k, \phi \rangle.$$

Then the computations above give

$$t\ddot{f} - (n-1)\dot{f} = t \int_{S(t)} \langle k, P_2(\phi) \rangle + (1 + t^2) \int_{S(t)} \langle k, P_1(\phi) \rangle$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} t^2 \langle k, P_0(\phi) \rangle + \frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} t \langle k, P_1(\phi) \rangle. \tag{4.25}$$

First observe that

$$\int_{S(t)} \langle k, \phi \rangle = o(t^n) \Rightarrow \int_{S(t)} \langle k, P_k(\phi) \rangle = o(t^n), \tag{4.26}$$

for all C^∞ forms ϕ vanishing to infinite order at ∂C. For if the left side of (4.26) holds, then $\int_{S(t)} \langle k, \partial^k \phi \rangle = o(t^n)$, since the hypotheses on ϕ are closed under differentiation. The coefficients of P_k are at least continuous, and it is elementary to verify that if $\int_{S(t)} \langle k, \partial^k \phi \rangle = o(t^n)$, then $\int_{S(t)} \langle k, \phi \partial^k \phi \rangle = o(t^n)$, for any function ϕ continuous on $\bar{C}$. Note that the same result holds with p in place of n, for any $p < \infty$.

It follows from (4.26) and the initial hypothesis (4.22) that the first two terms on the right in (4.25) are $o(t^n)$ as $t \to 0$. Since $t\ddot{f} - (n-1)\dot{f} = t^n \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\dot{f}}{t^{n-1}})$, this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{f}}{t^{n-1}} \right) = o(1) + t^{-n} \frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} t \langle k, P_1(\phi) \rangle + t^{-n} \frac{\mathrm{d}}{\mathrm{d}t} \int_{S(t)} t^2 \langle k, P_0(\phi) \rangle.$$

Integrating from 0 to t implies

$$\frac{\dot{f}}{t^{n-1}} = o(t) + t^{-n+1} \int_{S(t)} \langle k, P_1(\phi) \rangle + n \int_0^t t^{-n} \int_{S(t)} \langle k, P_1(\phi) \rangle + c_1 = o(t) + c_1,$$

where c_1 is a constant. A further integration using (4.26) again gives

$$f = o(t^{n+1}) + c_1' t^n + c_2, \tag{4.27}$$

where $c_1' = \frac{c_1}{n}$. Once more by (4.22), this implies that

$$f = o(t^{n+1}).$$

Note the special role played by the indicial root n here; if instead one had only $k = O(t^n)$, then the argument above does not give $k = O(t^{n+1})$ weakly.

This first estimate holds in fact for any given trace-free ϕ which is C^2 on $\bar{C}$, and vanishing to first order on ∂C. Working in the same way with the trace equation (4.20) shows that the same result holds for pure trace terms. In particular, it follows that

$$k = o(t^{n+1}) \text{ weakly.} \tag{4.28}$$

One now just repeats this argument inductively, with the improved estimate (4.28) in place of (4.22), using (4.26) inductively. Note that each inductive step requires higher differentiability of the test function ϕ and its higher order vanishing at ∂C.   ∎

Lemma 4.2 proves that $k = k_{\alpha\beta} = O(t^\nu)$ weakly, for any $\nu < \infty$. As discussed in Section 3, the transition from geodesic boundary coordinates to H-harmonic coordinates is $C^{2,\alpha}$ and hence

$$h = h_{\alpha\beta} = O(t^\nu), \tag{4.29}$$

weakly, with the level sets $S(t)$ replaced by S_τ. Next, as in Remark 3.5 and the proof of Theorem 3.1, the Eqs. (4.11)–(4.13) satisfy elliptic estimates, and elliptic regularity in weighted Hölder spaces, cf. [26,20], shows that the weak decay (4.29) implies strong or pointwise decay, i.e. (4.14) holds. The proof of Theorem 4.1 and thus Theorem 1.2 is now completed as before in the C^∞ smooth case.   ∎

Remark 4.3. In [3, Thm. 3.2], a proof of unique continuation of conformally compact Einstein metrics was given in dimension 4, using the fact that the compactified metric $\tilde{g}$ in (1.1) satisfies the Bach equation, together with the Calderón uniqueness theorem. However, the proof in [3] used harmonic coordinates; as discussed in Section 2, such coordinates do not preserve the Cauchy data. The first author is grateful to Robin Graham for pointing this out. Theorem 1.2 thus corrects this error, and generalizes the result to any dimension.

For the work to follow in Section 5, we note that Theorem 4.1 also holds for linearizations of the Einstein equations, i.e. forms k satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ric}_{g+tk} + n(g + tk))|_{t=0} = 0. \tag{4.30}$$

Thus, if k satisfies (4.30) and the analog of (4.7), i.e. $|k| = o(t^n)$, then k is pure gauge in Ω, in that $k = \delta^* Z$, where Z is a vector field on Ω with $Z = 0$ on $\partial\Omega$. The proof of this is exactly the same as the proof of Theorem 4.1, replacing the finite difference $k = g_1 - g_0$ by an infinitesimal difference.

This has the following consequence:

Corollary 4.4. *Let (M, g) be a conformally compact Einstein manifold with metric g having a $C^{3,\alpha}$ geodesic compactification. Suppose the topological condition (1.5) holds, i.e. $\pi_1(M, \partial M) = 0$.*

If k is an infinitesimal Einstein deformation on M as in (4.30), in divergence-free gauge, i.e.

$$\delta k = 0, \tag{4.31}$$

with $k = o(t^n)$ on approach to ∂M, then

$$k = 0 \quad on \ M.$$

Proof. The topological condition (1.5), together with the same analytic continuation argument at the end of the proof of Theorem 3.1, implies that k is pure gauge globally on M, in that $k = \delta^* Z$ on M with $Z = 0$ on ∂M. (Recall that (1.5) implies that ∂M is connected.) From (4.31), one then has

$$\delta\delta^* Z = 0,$$

on M. Pairing this with Z and integrating over $B(t)$, it follows that

$$\int_{B(t)} |\delta^* Z|^2 = \int_{S(t)} \delta^* Z(Z, N),$$

where N is the unit outward normal. Since $|Z|_g$ is bounded and $|\delta^* Z| vol(S(t)) = o(1)$ (since $|k| = o(t^n)$), it follows that

$$\int_M |\delta^* Z|^2 = 0,$$

which gives the result. ∎

Of course, analogs of these results also hold for bounded domains, via the proof of Theorem 3.1; the verification is left to the reader.

Remark 4.5. The analog of Proposition 3.7 most likely also holds in the setting of conformally compact metrics, for fields τ whose Euler–Lagrange equation is a diagonal system of Laplace-type operators to leading order, as in (3.40) or (3.41). The proof of this is basically the same as that of Proposition 3.7, using the proof of Theorem 1.2 and with the Mazzeo unique continuation result in place of that of Calderón. However, we will not carry out the details of the proof here.

5. Isometry extension and the constraint equations

In this section, we prove Theorem 1.3 that continuous groups of isometries at the boundary extend to isometries in the interior of complete conformally compact Einstein metrics and relate this issue in general to the constraint equations induced by the Gauss–Codazzi equations.

We begin with the following elementary consequence of Theorem 4.1.

Proposition 5.1. *Let* (Ω, g) *be a C^n polyhomogeneous conformally compact Einstein metric on a domain* $\Omega \simeq B^{n+1}$ *with boundary metric* γ *on* $\partial \Omega \simeq B^n$. *Suppose X is a Killing field on* $(\partial \Omega, \gamma)$ *and*

$$\mathcal{L}_X g_{(n)} = 0, \tag{5.1}$$

where $g_{(n)}$ is the nth term in the Fefferman–Graham expansion (4.3) *or* (4.4).
Then X extends to a Killing field on (Ω, g).

Proof. Extend X to a smooth vector field on Ω by requiring $[X, N] = 0$, where $N = \nabla \log t$ and t is the geodesic defining function determined by g and γ. Let ϕ_s be the corresponding 1-parameter group of diffeomorphisms and set $g_s = \phi_s^* g$. Then t is the geodesic defining function for g_s for any s, and the pair (g, g_s) satisfy the hypotheses of Theorem 4.1. Theorem 4.1 then implies that g_s is isometric to g, i.e. there exist diffeomorphisms ψ_s of Ω, equal to the identity on $\partial \Omega$, such that $\psi_s^* \phi_s^* g = g$. Thus $\phi_s \circ \psi_s$ is a 1-parameter group of isometries of g defined in Ω, with Y the corresponding Killing field. (In fact, $Y = X$, since any Killing field Y tangent to $\partial \Omega$ preserves the geodesics tangent to N, and so $[Y, N] = 0$. This determines Y uniquely in terms of its value at $\partial \Omega$. Since X satisfies the same equation with the same initial value, this gives the claim). ∎

We point out that the same result, and proof, also hold in the case of Einstein metrics on bounded domains, via Theorem 3.1; the condition (5.1) is of course replaced by $\mathcal{L}_X A = 0$. For some examples and discussion in the bounded domain case, see [1,2].

Suppose now that (M, g) is a (global) conformally compact Einstein metric and there is a domain Ω as in Proposition 5.1 contained in M on which (5.1) holds. Then by analytic continuation as discussed at the end of the proof of Theorem 3.1, X extends to a local Killing field on all of M, i.e. X extends to a Killing field on the universal cover $\widetilde{M}$. In particular, if the condition (1.5) holds, i.e. $\pi_1(M, \partial M) = 0$, then X extends to a global Killing field on M. Again, the same result holds in the context of bounded domains.

Remark 5.2. A natural analog of Proposition 5.1 holds for conformal Killing fields on $(\partial \Omega, \gamma)$, i.e. vector fields which preserve the conformal class $[\gamma]$ at conformal infinity. Such vector fields satisfy the conformal Killing equation

$$\hat{\mathcal{L}}_X \gamma = \mathcal{L}_X \gamma - \frac{tr(\mathcal{L}_X \gamma)}{n} \gamma = 0. \tag{5.2}$$

Namely, since we are working locally, it is well known – and easy to prove – that any non-vanishing conformal Killing field is Killing with respect to a conformally related metric $\widetilde{\gamma} = \lambda^2 \gamma$, so that

$$\mathcal{L}_X \widetilde{\gamma} = 0.$$

Hence, if $\mathcal{L}_X \widetilde{g}_{(n)} = 0$, then Proposition 5.1 implies that X extends to a Killing field on Ω.

One may express $\widetilde{g}_{(n)}$ in terms of λ and the lower order terms $g_{(k)}, k < n$ in the Fefferman–Graham expansion (4.3) and (4.4); however, the expressions become very complicated for n even and large, cf. [17]. Thus, while the Eq. (5.2) is conformally invariant, the corresponding conformally invariant equation for $g_{(n)}$ will be complicated in general.

Next we consider the constraint equations (4.5) in detail, i.e.

$$\delta \tau_{(n)} = 0 \quad \text{and} \quad tr\tau_{(n)} = a_{(n)}, \tag{5.3}$$

where $\tau_{(n)} = g_{(n)} + r_{(n)}$; $r_{(n)}$ and $a_{(n)}$ are explicitly determined by the boundary metric $\gamma = g_{(0)}$ and its derivatives up to order n. Both vanish when n is odd.

As will be seen below, the most important issue is the divergence constraint in (5.3), which arises from the Gauss–Codazzi equations. To see this, in the setting of Section 4, on $S(t) \subset (M, g)$, the Gauss–Codazzi equations are

$$\delta(A - Hg) = -\mathrm{Ric}(N, \cdot), \tag{5.4}$$

as 1-forms on $S(t)$; here $N = -t\partial_t$ is the unit outward normal. The same equation holds on a geodesic compactification $(M, \bar{g})$. If g is Einstein, then $\mathrm{Ric}(N, \cdot) = \bar{\mathrm{Ric}}(\bar{N}, \cdot) = 0$; the latter equality follows from (4.9). The Eq. (5.4) holds for all t small, and differentiating $(n - 1)$ times with respect to t gives rise to the divergence constraint in (5.3).

The Gauss–Codazzi equations are not used in the derivation and properties of the Fefferman–Graham expansion (4.3) and (4.4) per se. The derivation of these equations involves only the tangential (ij) part of the Ricci curvature. The asymptotic behavior of the normal (00) part of the Ricci curvature gives rise to the trace constraint in (5.3), cf. (4.19) and (4.20).

Let $\mathcal{T}$ be the space of pairs $(g_{(0)}, \tau_{(n)})$ satisfying (5.3). If $\tau_{(n)}^0$ is any fixed solution of (5.3), then any other solution with the same $g_{(0)}$ is of the form $\tau_{(n)} = \tau_{(n)}^0 + \tau$, where τ is transverse-traceless with respect to $g_{(0)}$. (Of course if n is odd, one may take $\tau_{(n)}^0 = 0$). The space $\mathcal{T}$ is naturally projects onto $\mathrm{Met}(\partial M)$ with fiber at γ an affine space of symmetric tensors and is a subset of the product $\mathrm{Met}(\partial M) \times \mathbb{S}^2(\partial M) \simeq T(\mathrm{Met}(\partial M))$. Let

$$\pi : \mathcal{T} \to \mathrm{Met}(\partial M) \tag{5.5}$$

be the projection onto the base space $\mathrm{Met}(\partial M)$ (the first factor projection).

By the discussion in Section 4, $(g_{(0)}, \tau_{(n)}) \in \mathcal{T}$ if and only if the corresponding pair $(g_{(0)}, g_{(n)})$ determine a formal polyhomogeneous solution to the Einstein equations near conformal infinity, i.e. formal series solutions containing log terms, as in (4.3) and (4.4). In fact, if $g_{(0)}$ and $g_{(n)}$ are real–analytic on ∂M, a result of Kichenassamy [24] implies that the series (4.3) or (4.4) converges, and gives an Einstein metric g, defined in a neighborhood of ∂M. The metric g is complete near ∂M and has a conformal compactification inducing the given data $(g_{(0)}, g_{(n)})$ on ∂M. Here we recall from the discussion in Section 4 that all coefficients of the expansion (4.3) or (4.4) are determined by $g_{(0)}$ and $g_{(n)}$.

In this regard, consider the following:

Problem. Is $\pi : \mathcal{T} \to \mathrm{Met}(\partial M)$ an open map? Thus, given any $(g_{(0)}, \tau_{(n)}) \in \mathcal{T}$ and any boundary metric $\widetilde{g}_{(0)}$ sufficiently close to $g_{(0)}$, does there exist $\widetilde{\tau}_{(n)}$ close to $\tau_{(n)}$ such that $(\widetilde{g}_{(0)}, \widetilde{\tau}_{(n)}) \in \mathcal{T}$.

Although π is obviously globally surjective, the problem above is whether π is locally surjective. For example, a simple fold map $x \to x^3 - x$ is not locally surjective near $\pm\sqrt{3}/3$. Observe that the trace condition in (5.3) imposes no constraint on $g_{(0)}$; given any $g_{(0)}$, it is easy to find $g_{(n)}$ such that $tr_{g_{(0)}}(g_{(n)} + r_{(n)}) = a_{(n)}$; this equation can readily be solved algebraically for many $g_{(n)}$.

By the inverse function theorem, it suffices (and is probably also necessary), to examine the problem above at the linearized level. However the linearization of the divergence condition in (5.3) gives a non-trivial constraint on the variation $h_{(0)}$ of $g_{(0)}$. Namely, the linearization in this case gives

$$\delta'(\tau_{(n)}) + \delta(\tau_{(n)})' = 0, \tag{5.6}$$

where $\delta' = \frac{\mathrm{d}}{\mathrm{d}u} \delta_{g_{(0)} + uh_{(0)}}$, and similarly for $(\tau_{(n)})'$.

Whether (5.6) is solvable for any $h_{(0)} \in S^2(\partial M)$ depends on the data $g_{(0)}$ and $g_{(n)}$. For example, it is trivially solvable when $\tau_{(n)} = 0$. For compact ∂M, one has

$$\Omega^1(\partial M) = \operatorname{Im}\delta \oplus \operatorname{Ker}\delta^*, \tag{5.7}$$

where Ω^1 is the space of 1-forms, so that solvability in general requires that

$$\delta'(\tau_{(n)}) \in \operatorname{Im}\delta = (\operatorname{Ker}\delta^*)^\perp. \tag{5.8}$$

Of course $\operatorname{Ker}\delta^*$ is exactly the space of Killing fields on $(\partial M, \gamma)$, and so this space serves as a potential obstruction space.

Clearly then π is locally surjective when $(\partial M, g_{(0)})$ has no Killing fields. On the other hand, it is easy to construct examples where $(\partial M, \gamma)$ does have Killing fields and π is not locally surjective:

Example 5.3. Let $(\partial M, g_{(0)})$ be the flat metric on the n-torus T^n, $n \geq 3$, and define $g_{(n)} = -(n-2)(\mathrm{d}\theta^2)^2 + (\mathrm{d}\theta^3)^2 + \cdots + (\mathrm{d}\theta^n)^2$. Then $g_{(n)}$ is transverse-traceless with respect to $g_{(0)}$. Let $f = f(\theta^1)$. Then $\hat{g}_{(n)} = fg_{(n)}$ is still transverse-traceless with respect to $g_{(0)}$, so that $(g_{(0)}, \hat{g}_{(n)}) \in \mathcal{T}$, at least for n odd.

It is then not difficult to see via a direct calculation, or more easily via Proposition 5.4, that (5.8) does not hold, so that π is not locally surjective.

Next we relate these two issues, i.e. the general solvability of the divergence constraint (5.8) and the extension of Killing fields on the boundary into the bulk. The following result holds for general $\phi \in S^2(\partial M)$ with $\delta\phi = 0$ on ∂M.

Proposition 5.4. *If X is a Killing field on $(\partial M, \gamma)$, with ∂M compact, then*

$$\int_{\partial M} \langle \mathcal{L}_X \tau_{(n)}, h_{(0)}\rangle \mathrm{d}V = -2\int_{\partial M}\langle \delta'(\tau_{(n)}), X\rangle \mathrm{d}V, \tag{5.9}$$

where $\delta' = \frac{\mathrm{d}}{\mathrm{d}s}\delta_{\gamma+sh_{(0)}}$. In particular, (5.1) holds for all Killing fields on $(\partial M, \gamma)$ if and only if the linearized divergence constraint vanishes, i.e. (5.8) holds.

Proof. Since X is a Killing field on $(\partial M, \gamma)$, one has

$$\int_{\partial M}\langle \mathcal{L}_X\tau, h\rangle \mathrm{d}V_\gamma = -\int_{\partial M}\langle \tau, \mathcal{L}_X h\rangle \mathrm{d}V_\gamma. \tag{5.10}$$

Setting $\gamma_s = \gamma + sh$, the divergence theorem gives

$$0 = \int_{\partial M}\delta_{\gamma_s}(\tau(X))\mathrm{d}V_{\gamma_s} = \int_{\partial M}\langle \delta_{\gamma_s}\tau, X\rangle \mathrm{d}V_{\gamma_s} - \frac{1}{2}\int_{\partial M}\langle \tau, \mathcal{L}_X\gamma_s\rangle \mathrm{d}V_{\gamma_s}, \tag{5.11}$$

where the second equality is a simple computation from the definitions; the inner products are with respect to γ_s. Taking the derivative with respect to s at $s = 0$, and using the facts that X is Killing and $\delta(\tau) = 0$, it follows that

$$\int_{\partial M}\langle \delta'\tau, X\rangle \mathrm{d}V - \frac{1}{2}\int_{\partial M}\langle \tau, \mathcal{L}_X h\rangle \mathrm{d}V = 0. \tag{5.12}$$

Combining this with (5.10) then gives (5.9); note that $\mathcal{L}_X r_{(n)} = 0$ in this case, since $r_{(n)}$ is determined by the boundary metric.

To prove the last statement, by (5.9), (5.1) holds if and only if $\int_{\partial M}\langle \delta'(\tau_{(n)}), X\rangle = 0$, for all variations h. If (5.8) holds, then $\delta'(\tau_{(n)}) = \delta h'_{(n)}$, for some $h'_{(n)}$ and so $\int_{\partial M}\langle \delta'(\tau_{(n)}), X\rangle = \int_{\partial M}\langle h'_{(n)}, \delta^*X\rangle = 0$, since X is Killing. The converse of this argument holds equally well. ∎

Proposition 5.4 implies that in general, Killing fields on ∂M do not extend to Killing fields in a neighborhood of ∂M (cf. Example 5.3). (Exactly the same result and proof hold in the bounded domain case, when the term $\tau_{(n)}$ is replaced by $A - Hg$.)

Now as noted above, whether isometry extension holds or not depends on the term $\tau_{(n)} = g_{(n)} + r_{(n)}$, or more precisely on the relation of the boundary metric $g_{(0)}$ with $\tau_{(n)}$. For Einstein metrics which are globally conformally compact, the term $\tau_{(n)}$ is determined, up to a finite dimensional moduli space, by the boundary metric $g_{(0)}$; (this is

discussed further below). Thus, whether isometry extension holds or not is quite a delicate issue; if so, it must depend crucially on the global structure of (M, g).

Before beginning the proof of Theorem 1.3, we first need to discuss some background material from [5,6].

Let E_{AH} be the space of conformally compact, or equivalently asymptotically hyperbolic Einstein metrics on M which have a C^∞ polyhomogeneous conformal compactification with respect to a fixed smooth defining function ρ, as in (1.1). In [5], it is shown that E_{AH} is a smooth, infinite dimensional manifold. One has a natural smooth boundary map

$$\Pi : E_{AH} \to \mathrm{Met}(\partial M), \tag{5.13}$$

sending g to its boundary metric γ.

The moduli space $\mathcal{E}_{AH}$ is the quotient $E_{AH}/\mathcal{D}_1$, where $\mathcal{D}_1$ is the group of smooth (polyhomogeneous) diffeomorphisms ϕ of M equal to the identity on ∂M. Thus, $g' \sim g$ if $g' = \phi^* g$, with $\phi \in \mathcal{D}_1$. Changing the defining function ρ in (1.1) changes the boundary metric conformally. Also, if $\phi \in \mathcal{D}_1$ then $\rho \circ \phi$ is another defining function, and all defining functions are of this form near ∂M. Hence if $\mathcal{C}$ denotes the space of smooth conformal classes of metrics on ∂M, then the boundary map (5.13) descends to a smooth map

$$\Pi : \mathcal{E}_{AH} \to \mathcal{C} \tag{5.14}$$

independent of the defining function ρ. The boundary map Π in (5.14) is Fredholm, of Fredholm index 0.

The linearization of the Einstein operator $\mathrm{Ric}_g + ng$ at an Einstein metric g is given by

$$\hat{L} = (\mathrm{Ric}_g + ng)' = \frac{1}{2} D^* D - R - \delta^* \beta, \tag{5.15}$$

acting on the space of symmetric 2-tensors $S^2(M)$ on M, cf. [10]. Here (as in Section 3), β is the Bianchi operator, $\beta(h) = \delta h + \frac{1}{2} dtrh$, Thus, $h \in T_g E_{AH}$ if and only if

$$\hat{L}(h) = 0.$$

The operator $\hat{L}$ is not elliptic, due to the $\delta^* \beta$ term. As is well known, this arises from the diffeomorphism group, and to obtain an elliptic linearization, one needs a gauge choice to break the diffeomorphism invariance of the Einstein equations. We will use a slight modification of the Bianchi gauge introduced in [11].

To describe this, given any fixed $g_0 \in E_{AH}$ with geodesic defining function t and boundary metric γ_0, let γ be a boundary metric near γ_0 and define the hyperbolic cone metric g_γ on γ by setting

$$g_\gamma = t^{-2}(dt^2 + \gamma);$$

g_γ is defined in a neighborhood of ∂M. Next, set

$$g(\gamma) = g_0 + \eta(g_\gamma - g_{\gamma_0}), \tag{5.16}$$

where η is a non-negative cut-off function supported near ∂M with $\eta = 1$ in a small neighborhood of ∂M. Any conformally compact metric g near g_0, with boundary metric γ then has the form

$$g = g(\gamma) + h, \tag{5.17}$$

where $|h|_{g_0} = O(t^2)$; equivalently $\bar{h} = t^2 h$ satisfies $\bar{h}_{ij} = O(t^2)$ in any smooth coordinate chart near ∂M. The space of such symmetric bilinear forms h is denoted by $\mathbb{S}_2(M)$ and the space of metrics g of the form (5.17) is denoted by Met_{AH}.

The Bianchi-gauged Einstein operator (with background metric g_0), is defined by

$$\Phi_{g_0} : \mathrm{Met}_{AH} \to \mathbb{S}_2(M) \tag{5.18}$$

$$\Phi_{g_0}(g) = \Phi(g(\gamma) + h) = \mathrm{Ric}_g + ng + (\delta_g)^* \beta_{g(\gamma)}(g),$$

where $\beta_{g(\gamma)}$ is the Bianchi operator with respect to $g(\gamma)$. By [11, Lemma I.1.4],

$$Z_{AH} \equiv \Phi^{-1}(0) \cap \{\mathrm{Ric} < 0\} \subset E_{AH}, \tag{5.19}$$

where {Ric < 0} is the open set of metrics with negative Ricci curvature. In fact, if $g \in E_{AH}$ is close to g_0, and $\Phi(g) = 0$, then $\beta_{g(\gamma)}(g) = 0$ and moreover

$$\delta_{g(\gamma)}(g) = 0 \quad \text{and} \quad tr_{g(\gamma)}(g) = 0. \tag{5.20}$$

The space Z_{AH} is a local slice for the action of $\mathcal{D}_1$ on E_{AH}: for any $g \in E_{AH}$ near g_0, there exists a diffeomorphism $\phi \in \mathcal{D}_1$ such that $\phi^* g \in Z_{AH}$, cf. again [11].

The linearization of Φ at $g_0 \in E_{AH}$ with respect to the second variable h has the simple form

$$(D_2 \Phi)_{g_0}(\dot{h}) = \frac{1}{2} D^* D \dot{h} - R_{g_0}(\dot{h}), \tag{5.21}$$

while the variation of Φ at g_0 with respect to the first variable $g(\gamma)$ has the form

$$(D_1 \Phi)_{g_0}(\dot{g}(\gamma)) = (D_2 \Phi)_{g_0}(\dot{g}(\gamma)) - \delta^*_{g_0} \beta_{g_0}(\dot{g}(\gamma)) = (\text{Ric}_g + ng)'(\dot{g}(\gamma)), \tag{5.22}$$

as in (5.15). Clearly $\dot{g}(\gamma) = \eta t^{-2} \dot{\gamma}$. The kernel of the elliptic self-adjoint linear operator

$$L = \frac{1}{2} D^* D - R \tag{5.23}$$

acting on the second variable h, represents the space of non-trivial infinitesimal Einstein deformations vanishing on ∂M. Let K denote the L^2 kernel of L. This is the same as the kernel of L on $\mathbb{S}_2(M)$, cf. [11,26]. An Einstein metric $g_0 \in E_{AH}$ is called *non-degenerate* if

$$K = 0. \tag{5.24}$$

For $g_0 \in \mathcal{E}_{AH}$ the kernel $K = K_{g_0}$ equals the kernel of the linear map $D\Pi : T_{g_0}\mathcal{E}_{AH} \to T_{\Pi(g_0)}\mathcal{C}$. Hence, g_0 is non-degenerate if and only if g_0 is a regular point of the boundary map Π in which case Π is a local diffeomorphism near g_0. From now on, we denote g_0 by g.

By the regularity result of Chruściel et al. [16], any $\kappa \in K$ has a C^∞ smooth polyhomogeneous expansion, analogous to the Fefferman–Graham expansion (4.3) and (4.4), with leading order terms satisfying

$$\kappa = O(t^n), \qquad \kappa(N, Y) = O(t^{n+1}), \qquad \kappa(N, N) = O(t^{n+1+\mu}), \tag{5.25}$$

where $N = -t \partial_t$ is the unit outward normal vector to the t-level set $S(t)$, Y is any g-unit vector tangent to $S(t)$ and $\mu > 0$; cf. also [28, Prop. 5]. Here $\kappa = O(t^n)$ means $|\kappa|_g = O(t^n)$. Also by an argument similar to the one leading to (5.20), any $\kappa \in K$ is transverse-traceless, i.e.

$$\delta\kappa = tr\kappa = 0. \tag{5.26}$$

Given this background, we are now ready to begin the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $\bar{g} = t^2 g$ be a geodesic compactification of g with boundary metric γ. By the boundary regularity result of [16], $\bar{g}$ is C^∞ polyhomogeneous on $\bar{M}$. It suffices to prove Theorem 1.3 for arbitrary 1-parameter subgroups of the isometry group of $(\partial M, \gamma)$. Thus, let ϕ_s be a local 1-parameter group of isometries of γ with $\phi_0 = id$, so that

$$\phi_s^* \gamma = \gamma.$$

The diffeomorphisms ϕ_s of ∂M may be extended to diffeomorphisms of M, so that the curve

$$g_s = \phi_s^* g \tag{5.27}$$

is a smooth curve in E_{AH}. By construction then, $\Pi[g_s] = [\gamma]$, so that $[h] = [\frac{dg_s}{ds}] \in \text{Ker } D\Pi$, for Π as in (5.14). One may then alter the diffeomorphisms ϕ_s by composition with diffeomorphisms in $\mathcal{D}_1$ if necessary, so that $h = \frac{dg_s}{ds} \in K_g$, where K_g is the kernel in (5.24). Denoting $h = \kappa$, it follows that

$$\kappa = \delta^* X, \tag{5.28}$$

where $X = d\phi_s/ds$ is smooth up to $\bar{M}$.

Thus it suffices to prove that $\delta^* X = 0$, since this will imply that $g_s = g$ (when g_s is modified by the action of $\mathcal{D}_1$). If $K_g = 0$, i.e. if g is a regular point of the boundary map Π, then this is now obvious (from the above), and proves the result in this special case; (the proof in this case requires only that (M, g) be $C^{2,\alpha}$ conformally compact).

We give two different (although related), proofs of Theorem 1.3, one conceptual and one more computational. The first, conceptual, proof involves an understanding of the cokernel of the map $D\Pi_g$ in $\mathrm{Met}(\partial M)$, and so one first needs to give an explicit description of this cokernel. To begin, recall the derivative

$$(D\Phi)_g : T_g \, \mathrm{Met}_{AH}(M) \to T_{\Phi(g)} \mathbb{S}_2(M). \tag{5.29}$$

Via (5.17), one has $T_g \, \mathrm{Met}_{AH} = T_\gamma \, \mathrm{Met}(\partial M) \oplus T_h \mathbb{S}_2(M)$ and the derivative with respect to the second factor is given by (5.21). If $K = 0$, then $D_2 \Phi$ is surjective at g (since $D_2 \Phi$ has index 0, and we recall that the kernel and cokernel here are equal to their L^2 counterparts), and hence so is $D\Pi$. In general, to understand $Coker \, D\Pi$, we show that $D\Phi$ is always surjective; this follows from the claim that for any non-zero $\kappa \in K$ there is a tangent vector $\dot{g}(\gamma) \in T_\gamma \, \mathrm{Met}(\partial M) \subset T_g \, \mathrm{Met}_{AH}$ such that

$$\int_M \langle (D_1 \Phi)_g(\dot{g}(\gamma)), \kappa \rangle \mathrm{d}V_g \neq 0. \tag{5.30}$$

Thus, the boundary variations $\dot{g}(\gamma)$ satisfying (5.30) for some κ correspond to the cokernel. To prove (5.30), let $B(t) = \{x \in M : t(x) \geq t\}$ and $S(t) = \partial B(t) = \{x \in M : t(x) = t\}$. Apply the divergence theorem to the integral (5.30) over $B(t)$; twice for the Laplace term in (5.22) and once for the δ^* term in (5.22). Since

$$\kappa \in \mathrm{Ker} \, L \quad \text{and} \quad \delta\kappa = 0,$$

it follows that the integral (5.30) reduces to an integral over the boundary, and gives

$$\int_{B(t)} \langle (D_1 \Phi)_g(\dot{g}(\gamma)), \kappa \rangle \mathrm{d}V_g = \frac{1}{2} \int_{S(t)} (\langle \dot{g}(\gamma), \nabla_N \kappa \rangle - \langle \nabla_N \dot{g}(\gamma), \kappa \rangle - 2\langle \beta(\dot{g}(\gamma)), \kappa(N) \rangle) \mathrm{d}V_{S(t)}. \tag{5.31}$$

Of course $\mathrm{d}V_{S(t)} = t^{-n} \mathrm{d}V_\gamma + O(t^{-(n-1)})$. By (5.25) the last term in (5.31) is then $O(t)$ and so may be ignored. Let

$$\widetilde{\kappa} = t^{-n}\kappa, \tag{5.32}$$

so that by (5.25), $|\widetilde{\kappa}|_g|_{S(t)} \leq C$. Setting $\hat{\kappa} = t^2 \widetilde{\kappa}$, one has $|\hat{\kappa}|_{\bar{g}} = |\widetilde{\kappa}|_g$, and so the same is true for $|\hat{\kappa}|_{\bar{g}}$. From the definition (5.16), a straightforward computation shows that near ∂M,

$$\dot{g}(\gamma) = t^{-2}\dot{\gamma}, \quad \text{and} \quad \nabla_N \dot{g}(\gamma) = 0.$$

Note that $|\dot{g}(\gamma)|_g \sim 1$ as $t \to 0$. Hence,

$$
\begin{aligned}
(\langle \dot{g}(\gamma), \nabla_N \kappa \rangle - \langle \nabla_N \dot{g}(\gamma), \kappa \rangle)_g \mathrm{d}V_{S(t)} &= t^2 \langle \nabla_N \kappa, \dot{\gamma} \rangle_\gamma \mathrm{d}V_{S(t)} + O(t) \\
&= \langle \nabla_N \hat{\kappa} - (n-2)\hat{\kappa}, \dot{\gamma} \rangle_\gamma \mathrm{d}V_\gamma + O(t).
\end{aligned}
$$

Thus,

$$\int_{B(t)} \langle (D_1 \Phi)_g(\dot{g}(\gamma)), \kappa \rangle \mathrm{d}V_g = \frac{1}{2} \int_{S(t)} \langle \nabla_N \hat{\kappa} - (n-2)\hat{\kappa}, \dot{\gamma} \rangle_\gamma \mathrm{d}V_\gamma + O(t). \tag{5.33}$$

Now suppose (contrary to (5.30)),

$$\nabla_N \hat{\kappa} - (n-2)\hat{\kappa} = O(t), \tag{5.34}$$

as forms on $(S(t), \bar{g})$; note however that ∇ is taken with respect to g in (5.34). It follows from the smooth polyhomogeneity of $\hat{\kappa}$ near ∂M and elementary integration that (5.34) gives

$$\kappa = o(t^n). \tag{5.35}$$

The form κ is an infinitesimal Einstein deformation, divergence-free by (5.26). Thus Corollary 4.4 (cf. also [5, Thm. 3.1]), and (5.35), together with the assumption in Theorem 1.3 that $\pi_1(M, \partial M) = 0$, imply that

$$\kappa = 0 \quad \text{on } M,$$

giving a contradiction. This proves the relation (5.30).

The proof above shows that the form

$$\dot{g}(\gamma) = \lim_{t \to 0} \hat{\kappa}|_{S(t)}, \tag{5.36}$$

on ∂M satisfies (5.30). The limit here exists by the smooth polyhomogeneity of κ at ∂M. Thus, the space

$$\hat{K} = \{\hat{\kappa} = \lim_{t \to 0} t^{-(n-2)} \kappa|_{S(t)} : \kappa \in K\}, \tag{5.37}$$

is naturally identified with the cokernel of $D\Pi_g$ in $T_\gamma \operatorname{Met}(\partial M)$. Note that $\dim \widetilde{K} = \dim K$ and also that the estimates (5.25) show that $\hat{\kappa} = \hat{\kappa}^T$ on ∂M. This means that infinitesimal deformations of the boundary metric γ in the direction $\hat{\kappa}$, $\hat{\kappa} \in \hat{K}$, are not realized as $\frac{d}{ds}\Pi(g_s)|_{s=0}$, where g_s is a curve in E_{AH} through g, i.e. a curve of *global* Einstein metrics on M. On the other hand, suppose that $\kappa = \delta^* X$, i.e. (5.28) holds for some $\kappa \in K$ and vector field X on M (necessarily) inducing a Killing field on $(\partial M, \gamma)$. Consider the local curve of metrics

$$g_s = g + s\delta^* \left(\frac{X}{t^n}\right) \tag{5.38}$$

defined in a neighborhood of ∂M. The curve g_s is Einstein to 1st order in s at $s = 0$. The induced variation of the boundary metric on $S(t)$ is, by construction, $(\widetilde{\kappa})^T|_{S(t)} \sim \widetilde{\kappa}|_{S(t)}$, which, by rescaling, compactifies to $\hat{\kappa}$ at ∂M; here $\widetilde{\kappa}$ is given as in (5.32). Now note that the linearized divergence constraint (5.6) or (5.8) only involves the behavior at ∂M, or equivalently, the limiting behavior on $(S(t), \gamma_t)$, $\gamma_t = \bar{g}|_{S(t)}$, as $t \to 0$. This basically shows that the constraint (5.6) may be solved in the direction $h_{(0)} = \hat{\kappa}$; a complete justification of this is given in the more computational proof to follow. Also, a simple calculation, cf. (5.42) below, gives $\mathcal{L}_X \tau_{(n)} = \mathcal{L}_X g_{(n)} = 2\hat{\kappa}$. (The first statement follows since the term $r_{(n)}$ is intrinsic to the boundary metric γ, so that $\mathcal{L}_X r_{(n)} = 0$). Hence, it follows from Proposition 5.4 that

$$2\int_{\partial M} |\hat{\kappa}|^2 dV_\gamma = \int_{\partial M} \langle \mathcal{L}_X \tau_{(n)}, \hat{\kappa}\rangle dV_\gamma = 2\int_{\partial M} \langle \delta(\tau'_{(n)}), X\rangle dV_\gamma = 2\int_{\partial M} \langle \tau'_{(n)}, \delta^* X\rangle dV_\gamma = 0, \tag{5.39}$$

and thus $\mathcal{L}_X \tau_{(n)} = 0$ on $(\partial M, \gamma)$. Corollary 4.4 or Proposition 5.1 and the assumption $\pi_1(M, \partial M) = 0$ then imply that $\kappa = 0$ on M, so that X is a Killing field on M. This completes the first proof of Theorem 1.3. ∎

From the converse part of Proposition 5.4, one also obtains:

Corollary 5.5. *Let g be a conformally compact Einstein metric on a compact manifold M with C^∞ boundary metric γ. Then the linearized divergence constraint equation (5.6) is always solvable on $(\partial M, \gamma)$, i.e. the map π in (5.5) is locally surjective at $(\gamma, \tau_{(n)})$.*

It is useful and of interest to give another, direct computational proof of Theorem 1.3, without using the identification (5.37) as the cokernel of $D\Pi$. The basic idea is to compute as in Proposition 5.4 on $(S(t), g_t)$, with $A - Hg_t$ in place of $\tau_{(n)}$, and then pass to the limit on ∂M. Throughout the proof, we assume (5.28) holds.

Before starting the proof per se, we note that the estimates (5.25) and (5.28) imply that X is tangential, i.e. tangential to $(S(t), g)$, to high order, in that

$$\langle X, N\rangle = O(t^{n+1+\mu}). \tag{5.40}$$

To see this, one has $(\delta^* X)(N, N) = \langle \nabla_N X, N\rangle = N\langle X, N\rangle$. Thus (5.40) follows from (5.25) and the claim that $\langle X, N\rangle = 0$ on ∂M. To prove the latter, consider the compactified metric $\bar{g} = t^2 g$. One has $\mathcal{L}_X \bar{g} = \mathcal{L}_X(t^2 g) = 2\frac{X(t)}{t}\bar{g} + O(t^n)$. Thus for the induced metric γ on ∂M, $\mathcal{L}_X \gamma = 2\lambda\gamma$, where $\lambda = \lim_{t \to 0} \frac{X(t)}{t}$. Since X is a Killing field on $(\partial M, \gamma)$, this gives $\lambda = 0$, which is equivalent to the statement that $\lim_{t \to 0}\langle X, N\rangle_g = 0$. Note also that since X is smooth up to ∂M, $|X|_g = O(t^{-1})$.

We claim also that

$$[X, N] = O(t^{n+1}), \tag{5.41}$$

in norm. First, $\langle [X, N], N\rangle = \langle \nabla_X N - \nabla_N X, N\rangle = -(\delta^* X)(N, N) = O(t^{n+1+\mu})$. On the other hand, on tangential g-unit vectors Y, $\langle [X, N], Y\rangle = \langle \nabla_X N - \nabla_N X, Y\rangle \sim \langle \nabla_X N, Y\rangle - 2(\delta^* X)(N, Y) + \langle \nabla_Y X, N\rangle \sim -2(\delta^* X)(N, Y) = O(t^{n+1})$, as claimed. Here $\sim$ denotes equality modulo terms of order $o(t^n)$. We have also used the fact that $\langle \nabla_X N, Y\rangle + \langle \nabla_Y X, N\rangle \sim X\langle N, Y\rangle = 0$.

Now, to begin the proof itself (assuming (5.28)), as above write

$$g_s = g + s\kappa + O(s^2) = g + s\delta^* X + O(s^2).$$

If t_s is the geodesic defining function for g_s (with boundary metric γ), then the Fefferman–Graham expansion gives $\bar{g}_s = \mathrm{d}t_s^2 + (\gamma + t_s^2 g_{(2),s} + \cdots + t_s^n g_{(n),s}) + O(t^{n+1})$. The estimate (5.40) implies that $t_s = t + s\,O(t^{n+2+\alpha}) + O(s^2)$, so that modulo lower order terms, we may view $t_s \sim t$. Taking the derivative of the FG expansion with respect to s at $s = 0$, and using the fact that X is Killing on $(\partial M, \gamma)$, together with the fact that the lower order terms $g_{(k)}, k < n$, are determined by γ, it follows that, for $\hat{k}$ as in (5.37),

$$\hat{\kappa} = \frac{1}{2}\mathcal{L}_X g_{(n)}, \tag{5.42}$$

at ∂M. Here both $\hat{\kappa}$ and $\mathcal{L}_X g_{(n)}$ are viewed as forms on $(\partial M, \gamma)$.

Next, we claim that on $(S(t), g_t)$,

$$\mathcal{L}_X A = -\frac{n-2}{2}t^{n-2}\mathcal{L}_X g_{(n)} + O(t^{n-1}). \tag{5.43}$$

To see this, one has $A = \frac{1}{2}\mathcal{L}_N g = -\frac{1}{2}\mathcal{L}_{t\partial t} g = -\frac{1}{2}\mathcal{L}_{t\partial t}(t^{-2}g_t)$. But $\mathcal{L}_{t\partial t}(t^{-2}g_t) = \sum \mathcal{L}_{t\partial t}(t^{-2+k} g_{(k)}) = \sum (k-2)t^{k-2}g_{(k)}$. The same reasoning as before then gives (5.43).

Given these results, we now compute

$$\int_{S(t)} \langle \mathcal{L}_X(A - Hg_t), \widetilde{\kappa} \rangle_{g_t}\,\mathrm{d}V_{S(t)};$$

compare with the left side of (5.9). First, by (5.43),

$$\int_{S(t)} \langle \mathcal{L}_X A, \widetilde{\kappa} \rangle_{g_t}\,\mathrm{d}V_{S(t)} = -\frac{n-2}{2}\int_{S(t)} \langle \mathcal{L}_X g_{(n)}, \hat{\kappa} \rangle_\gamma\,\mathrm{d}V_\gamma + O(t).$$

Next, one has $\mathcal{L}_X(Hg_t) = X(H)g_t + H\mathcal{L}_X g_t$. For the first term, $X(H) = tr\mathcal{L}_X A + O(t^n) = -\frac{n-2}{2}t^{n-2}tr\mathcal{L}_X g_{(n)} + O(t^n)$. Since $trg_{(n)}$ is intrinsic to γ and X is Killing on $(\partial M, \gamma)$, it follows that $X(H) = O(t^{n-1})$. Also, $\langle g_t, \widetilde{\kappa} \rangle = tr^T\widetilde{\kappa}$, where tr^T is the tangential trace. By (5.25) and the fact that κ is trace-free, $\langle g_t, \widetilde{\kappa} \rangle = O(t^{1+\alpha})$. Hence $X(H)\langle g_t, \widetilde{\kappa} \rangle\mathrm{d}V_{S(t)} = O(t^\alpha)$. Similarly, from (5.41) one computes $\mathcal{L}_X g_t = \mathcal{L}_X g + O(t^{n+1}) = 2t^n\widetilde{\kappa} + O(t^{n+1})$. Since $H \sim n$, using (5.42) this gives

$$-\int_{S(t)} \langle \mathcal{L}_X(Hg_t), \widetilde{\kappa} \rangle\mathrm{d}V_{S(t)} = -n\int_{S(t)} \langle \mathcal{L}_X g_{(n)}, \hat{\kappa} \rangle_\gamma\mathrm{d}V_\gamma + O(t^\alpha).$$

Combining these computations then gives

$$\int_{S(t)} \langle \mathcal{L}_X(A - Hg_t), \widetilde{\kappa} \rangle_{g_t}\mathrm{d}V_{S(t)} = -\left(\frac{n-2}{2} + n\right)\int_{\partial M} \langle \mathcal{L}_X g_{(n)}, \hat{\kappa} \rangle_\gamma\mathrm{d}V_\gamma + o(1). \tag{5.44}$$

On the other hand, one may use the method of proof of Proposition 5.4 to compute the left side of (5.44). First since on $S(t)$, $\tau = A - Hg_t$ is divergence-free, a slight extension of the calculation (5.9) gives, for any vector field Y tangent to $S(t)$ and variation h of $g_t = g|_{s(t)}$ on $S(t)$,

$$\int_{S(t)} \langle \mathcal{L}_Y \tau, h \rangle_{g_t}\mathrm{d}V_{g_t} = -2\int_{S(t)} \langle \delta'(\tau), Y \rangle\mathrm{d}V_{g_t} + \int_{S(t)} [\delta Y\langle \tau, h \rangle + \langle \tau, \delta^* Y \rangle trh]\mathrm{d}V_{g_t}. \tag{5.45}$$

Now let the tangential variation h be given by $h = (\partial^* \frac{X}{t^n})^T$ where $\delta^* = \partial_g^*$. Thus, $h = (t^{-n}\kappa)^T = (\widetilde{\kappa}^T)$, for $\widetilde{\kappa}$ as in (5.32). Also, set $Y = X^T$. Observe that the estimate (5.40) implies that X agrees with X^T to high degree, in that $X = X^T + O^{(n+1+\mu)}$. This has the effect that one may use X and X^T interchangably in the computations below. For example, since κ is trace-free, $\delta_g X = 0$ and hence a simple calculation shows that $\delta_{g_t} Y = O(t^{n+1+\mu})$. Similarly, $tr^T h = O(t^{1+\mu})$, while $\delta^* Y = O(t^n)$. In particular, the second term on the right in (5.45) is $O(t)$, and hence may be ignored.

Next, the deformation h above is the tangential part of a (trivial) infinitesimal Einstein deformation, and hence the linearized divergence constraint (5.6) holds along $S(t)$, in the direction h. Arguing then as in the proof of Proposition 5.4, it follows from (5.45) and the fact that $X^T \sim X$ to high order that

$$\int_{S(t)} \langle \mathcal{L}_X(A - Hg_t), \tilde{\kappa} \rangle_{gt} \mathrm{d}V_{S(t)} = 2 \int_{S(t)} \langle (A - Hg_t)', (\delta^* X)^T \rangle \mathrm{d}V_{gt} + O(t). \tag{5.46}$$

Now $A' = \frac{\mathrm{d}}{\mathrm{d}s}(A_{g+s\tilde{\kappa}}) = \frac{1}{2}(\mathcal{L}_N \tilde{\kappa} + \mathcal{L}_{N'} g) = \frac{1}{2}\nabla_N \tilde{\kappa} + \tilde{\kappa} + O(t)$. Similarly, $(Hg_t)' = H'g_t + H(gt)'$. The first term here, when paired with $(\delta^* X)^T$ and integrated, gives $O(t)$, while the second term is $n\tilde{\kappa}$ to leading order. Thus one has

$$2 \int_{S(t)} \langle (A - Hg_t)', (\delta^* X)^T \rangle \mathrm{d}V_{gt} = \int_{S(t)} \langle \nabla_N \tilde{\kappa} - (2n - 2)\tilde{\kappa}, \kappa \rangle_{gt} \mathrm{d}V_{gt} + O(t). \tag{5.47}$$

A straightforward calculation, essentially the same as that preceding (5.33) shows that

$$\int_{S(t)} \langle \nabla_N \tilde{\kappa} - (2n - 2)\tilde{\kappa}, \kappa \rangle_{gt} \mathrm{d}V_{gt} = \int_{S(t)} \left[\frac{1}{2} N(|\hat{\kappa}|^2) - (2n - 2)|\hat{\kappa}|^2 \right] \mathrm{d}V_\gamma + O(t), \tag{5.48}$$

where the norms on the right are with respect to $\bar{g}$. The first term on the right in (5.48) is $O(t)$, and comparing (5.48) with (5.44)–(5.47) shows that

$$\hat{\kappa} = 0$$

on ∂M. Hence via Corollary 4.4, $\kappa = 0$ on M, as before. This completes the second proof of Theorem 1.3. ∎

Proof of Corollary 1.4. Suppose (M, g) is a conformally compact Einstein metric with boundary metric given by the round metric $S^n(1)$ on S^n. Theorem 1.3 implies that the isometry group of (M, g) contains the isometry group of S^n. This reduces the Einstein equations to a simple system of ODEs, and it is easily seen that the only solution is given by the Poincaré metric on the ball B^{n+1}. ∎

Remark 5.6. By means of Obata's theorem [30], Theorem 1.3 remains true for continuous groups of conformal isometries at conformal infinity. Thus, the class of the round metric on S^n is the only conformal class which supports a non-essential conformal Killing field, i.e. a field which is not Killing with respect to some conformally related metric. Corollary 1.4 shows that any $g \in E_{AH}$ with boundary metric $S^n(1)$ is necessarily the hyperbolic metric g_{-1} on the ball. For g_{-1}, it is well known that essential conformal Killing fields on S^n extend to Killing fields on $(\mathbb{H}^{n+1}, g_{-1})$.

We expect that a modification of the proof of Theorem 1.3 would give this result directly, without the use of Obata's theorem. In fact, such would probably give (yet) another proof of Obata's result.

Corollary 5.5 shows, in the global situation, that the projection π of the constraint manifold $\mathcal{T}$ to Met(∂M) is always locally surjective. Hence there exists a formal solution, and an exact solution in the analytic case, for any nearby boundary metric, which is defined in a neighborhood of the boundary. However, the full boundary map Π in (5.13) or (5.14) on global metrics is not locally surjective in general; nor is it always globally surjective.

The simplest example of this behavior is provided by the family of AdS Schwarzschild metrics. These are metrics on $\mathbb{R}^2 \times S^{n-1}$ of the form

$$g_m = V^{-1}\mathrm{d}r^2 + V\mathrm{d}\theta^2 + r^2 g_{S^{n-1}(1)},$$

where $V = V(r) = 1 + r^2 - \frac{2m}{r^{n-2}}$. Here $m > 0$ and $r \in [r_+, \infty]$, where r_+ is the largest root of the equation $V(r_+) = 0$. The locus $\{r_+ = 0\}$ is a totally geodesic round S^{n-1} of radius r_+. Smoothness of the metric at $\{r_+ = 0\}$ requires that the circular parameter θ runs over the interval $[0, \beta]$, where

$$\beta = \frac{4\pi r_+}{nr_+^2 + (n - 2)}.$$

The metrics g_m are isometrically distinct for distinct values of m, and form a curve in E_{AH} with conformal infinity given by the conformal class of the product metric on $S^1(\beta) \times S^{n-1}(1)$. As m ranges over the interval $(0, \infty)$, β has

a maximum value of

$$\beta \leq \beta_{\max} = 2\pi\sqrt{(n-2)/n}.$$

As $m \to 0$ or $m \to \infty$, $\beta \to 0$.

Hence, the metrics $S^1(L) \times S^{n-1}(1)$ are not in $\Pi(g_m)$ for any $L > \beta_{\max}$. In fact these boundary metrics are not in $\text{Im}(\Pi)$ generally, for any manifold M^{n+1}. For Theorem 1.3 implies that any conformally compact Einstein metric with boundary metric $S^1(L) \times S^{n-1}(1)$ has an isometry group containing the isometry group of $S^1(L) \times S^{n-1}(1)$. This again reduces the Einstein equations to a system of ODEs and it is easy to see (although we do not give the calculations here), that any such metric is an AdS Schwarzschild metric.

Remark 5.7. In the context of Propositions 5.1 and 5.4, it is natural to consider the issue of whether local Killing fields of ∂M (i.e. Killing fields defined on the universal cover), extend to local Killing fields of any global conformally compact Einstein metric. Note that Propositions 5.1 and 5.4 are both local results, the latter by using variations $h_{(0)}$ which are of compact support. However, the linearized constraint condition (5.8) is not invariant under covering spaces; even the splitting (5.7) is not invariant under coverings, since a Killing field on a covering space need not descend to the base space.

We claim that local Killing fields do not extend even locally into the interior in general. As a specific example, let N^{n+1} be any complete, geometrically finite hyperbolic manifold, with conformal infinity $(\partial N, \gamma)$, and which has at least one parabolic end, i.e. a finite volume cusp end, with cross sections given by flat tori T^n. There exist many such manifolds. The metric at conformal infinity is conformally flat, so there are many local Killing fields on ∂N. For example, in many examples N itself is a compact hyperbolic manifold. Of course the local (conformal) isometries of ∂N extend here to local isometries of N.

However, as shown in [15], the cusp end may be capped off by Dehn filling with a solid torus, to give infinitely many distinct conformally compact Einstein metrics with the same boundary metric $(\partial N, \gamma)$. These Dehn-filled Einstein metrics cannot inherit all the local conformal symmetries of the boundary.

Remark 5.8. We point out that Theorem 1.3 fails for complete Ricci-flat metrics which are ALE (asymptotically locally Euclidean). The simplest counterexamples are the family of Eguchi–Hanson metrics, which have boundary metric at infinity given by the round metric on $S^3/\mathbb{Z}_2$. The symmetry group of these metrics is strictly smaller than the isometry group $\text{Isom}(S^3/\mathbb{Z}_2)$ of the boundary. Similarly, the Gibbons–Hawking family of metrics with boundary metric the round metric on $S^3/\mathbb{Z}_k$ have only an S^1 isometry group, much smaller than the group $\text{Isom}(S^3/\mathbb{Z}_k)$.

This indicates that, despite a number of proposals, some important features of holographic renormalization in the AdS context cannot carry over to the asymptotically flat case.

Acknowledgements

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